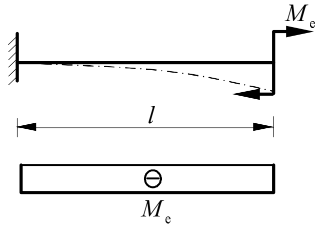
第五章　梁弯曲时的位移

5-1　试用积分法验算附录Ⅳ中第 1 至第 8 项各梁的挠曲线方程及最大挠度、梁端转角的表达式。

解：（1）$M=-M_{\mathrm{e}} \qquad (0\le x\le l)$



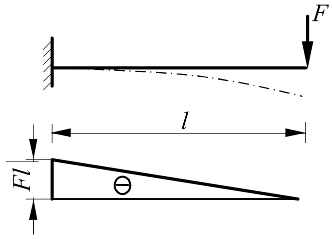
$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}=-\frac{-M_{\mathrm{e}}}{EI}$$

$$\frac{\mathrm{d}w}{\mathrm{d}x}=\frac{M_{\mathrm{e}}}{EI}x+C, w=\frac{M_{\mathrm{e}}}{2EI}x^2+Cx+D$$

$$\theta(0)=\frac{\mathrm{d}w}{\mathrm{d}x}=0, C=0, w(0)=0, D=0$$

得　$w=\dfrac{M_{\mathrm{e}}}{2EI}x^2$，$w_{\max}=\dfrac{M_{\mathrm{e}}l^2}{2EI}, \theta_B=\dfrac{M_{\mathrm{e}}l}{EI}$

（2）$M=-Fl+Fx \qquad (0\le x\le l)$



$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}=-\frac{-Fl+Fx}{EI}$$

$$\frac{\mathrm{d}w}{\mathrm{d}x}=-\frac{1}{EI}(-Flx+\frac{1}{2}Fx^2)+C$$

$$w=-\frac{1}{EI}(-\frac{1}{2}Flx^2+\frac{1}{6}Fx^3)+Cx+D$$

$$\theta(0)=0, C=0; w(0)=0, D=0$$

故

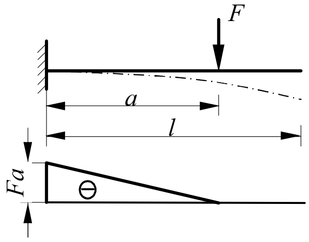
$$w=\frac{Fx^2}{6EI}(3l-x)$$

$$\theta_B=-\frac{1}{EI}(-Fl^2+\frac{1}{2}Fl^2)=\frac{Fl^2}{2EI}$$

$$w_B=\frac{Fl^3}{3EI}$$

（3）$M=-Fa+Fx \qquad (0\le x\le a)$

由（2）解　$w=\dfrac{Fx^2}{6EI}(3a-x) \qquad (0\le x\le a)$



$$w(a)=\frac{Fa^3}{3EI}$$

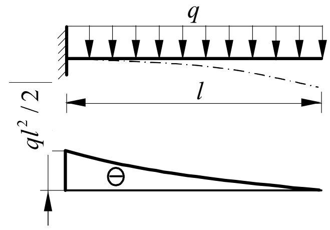
$$\theta_C=\frac{F}{6EI}(6ax-3x^2)\Big|_{x=a}=\frac{Fa^2}{2EI}$$

$$\theta_B=\theta_C=\frac{Fa^2}{2EI}$$

$$w=\frac{Fa^3}{3EI}+\frac{Fa^2}{2EI}(x-a)=\frac{Fa^2}{6EI}(2a+3x-3a)=\frac{Fa^2}{6EI}(3x-a) \quad (a\le x\le l)$$

$$w_B=w(l)=\frac{Fa^2}{6EI}(3l-a)$$

（4）$M=-\frac{1}{2}ql^2+qlx-\frac{1}{2}qx^2$



$$\frac{d^2w}{dx^2}=-\frac{M}{EI}$$

$$\frac{d^2w}{dx^2}=\frac{1}{EI}(\frac{1}{2}ql^2-qlx+\frac{1}{2}qx^2)$$

$$\frac{d^2w}{dx^2}=\frac{1}{EI}(\frac{1}{2}ql^2x-qlx^2+\frac{1}{6}qx^3)+C$$

$$\theta(0)=0, C=0, \theta_B=\frac{ql^3}{6EI}$$

$$w=\frac{1}{EI}(\frac{1}{4}ql^2x^2-\frac{1}{6}qlx^3+\frac{1}{24}qx^4)+Cx+D$$

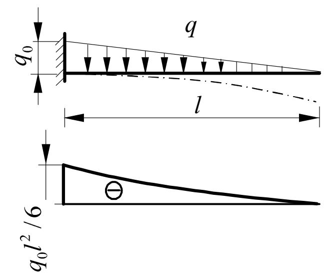
$$w(0)=0, D=0$$

$$w=\frac{qx^2}{24EI}(6l^2-4lx+x^2), w_B=\frac{ql^4}{8EI}$$

（5）$q(x)=q_0(1-\frac{x}{l})$

$$M=-\frac{1}{6}q_0l^2+\frac{1}{2}q_0lx-q(x)\cdot x\cdot\frac{x}{2}-\frac{1}{2}x[q_0-q(x)]\cdot\frac{2}{3}x$$

$$=\frac{-q_0}{6l}(l^3-3l^2x+3lx^2\cdot x^3)$$



$$\frac{d^2w}{dx^2}=-\frac{M}{EI}$$

$$\frac{dw}{dx}=\frac{q_0}{6lEI}(l^3x-\frac{3}{2}l^2x^2+lx^3-\frac{1}{4}x^4)+C$$

$$\theta(0)=0, C=0, \theta_B=\frac{q_0l^3}{24EI}$$

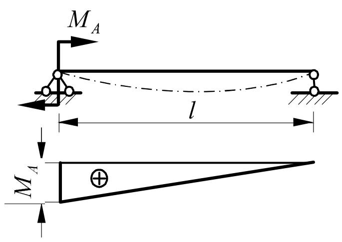
$$w=\frac{q_0}{6lEI}(\frac{1}{2}l^3x^2-\frac{1}{2}l^2x^3+\frac{1}{4}lx^4-\frac{1}{20}x^5)+Cx+D$$

$$w(0)=0, D=0$$

$$w=\frac{qx^2}{120lEI}(10l^3-10l^2x+5lx^2-x^3)$$

$$w_B=\frac{ql^4}{30EI}$$

（6）$M=M_A-\frac{M_A}{l}x$

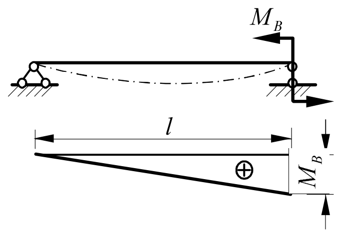


$$\frac{d^2w}{dx^2}=-\frac{1}{EI}(M_A-\frac{M_A}{l}x)$$

$$\frac{d^2w}{dx^2}=-\frac{M_A}{EI}(1-\frac{x}{l})$$

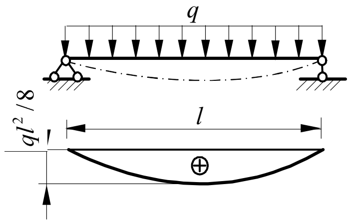
$$\frac{\mathrm{d}w}{\mathrm{d}x}=\frac{M_A}{EI}(\frac{x^2}{2l}-x)+C$$
$$w=\frac{M_A}{EI}(\frac{x^3}{6l}-\frac{x^2}{2})+Cx+D$$
$$w(0)=0,D=0$$
$$w(l)=0,\frac{M_A}{EI}(\frac{l^2}{6}-\frac{l^2}{2})+Cl=0$$
$$C=\frac{M_A l}{3EI}$$
$$w=\frac{M_A}{EI}(\frac{x^3}{6l}-\frac{x^2}{2})+\frac{M_A l}{3EI}x$$
$$=\frac{M_A x}{6EIl}(2l^2+x^2-3lx),w_C=w(\frac{l}{2})=\frac{M_A l^2}{16EI}$$
$$\theta=\frac{\mathrm{d}w}{\mathrm{d}x}=\frac{M_A}{6EIl}(2l^2+3x^2-6lx)$$
$$\theta_A=\frac{M_A l}{3EI},\theta_B=-\frac{M_A l}{6EI}$$

（7）$M=\dfrac{M_B}{l}x$



$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}=-\frac{M}{EI}$$
$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}=-\frac{M_B}{EIl}x$$
$$\frac{\mathrm{d}w}{\mathrm{d}x}=-\frac{M_B}{2EIl}x^2+C$$
$$w=-\frac{M_B x^3}{6EIl}+Cx+D$$
$$w(0)=0,D=0$$
$$w(l)=-\frac{M_B l^3}{6EIl}+Cl=0$$
$$C=\frac{M_B l}{6EI}$$
$$w=-\frac{M_B x^3}{6EIl}+\frac{M_B l}{6EI}x=\frac{M_B x}{6EIl}(l^2-x^2)$$
$$w_C=w(\frac{l}{2})=\frac{M_B l^2}{16EI}$$
$$\theta=\frac{\mathrm{d}w}{\mathrm{d}x}=\frac{M_B}{6EIl}(l^2-3x^2)$$
$$\theta_A=\theta(0)=\frac{M_B l}{6EI},\quad \theta_B=\theta(l)=-\frac{M_B l}{3EI}$$

（8）$M=\dfrac{ql}{2}x-\dfrac{1}{2}qx^2$



$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}=-\frac{M}{EI}$$

$$EI\frac{\mathrm{d}w}{\mathrm{d}x}=\frac{ql}{4}x^2-\frac{q}{6}x^3+C$$

$$EIw=\frac{ql}{12}x^3-\frac{q}{24}x^4+Cx+D$$

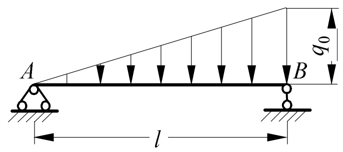
$$w(0)=0,D=0;w(l)=0,\frac{ql}{12}\cdot l^3-\frac{q}{24}l^4+Cl=0,C=\frac{-ql^3}{24}$$

$$w=\frac{-1}{EI}(\frac{ql}{12}x^3-\frac{q}{24}x^4-\frac{ql^3}{24}x)=\frac{qx}{24EI}(l^3-2lx^2+x^3)$$

$$\theta=\frac{\mathrm{d}w}{\mathrm{d}x}=\frac{q}{24EI}(l^3-6lx^2+4x^3)$$

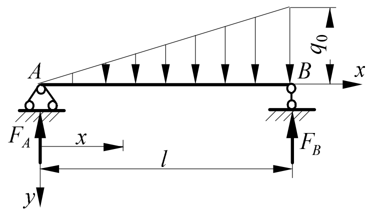
$$\theta_A=\frac{ql^3}{24EI},\theta_B=\theta(l)=-\frac{ql^3}{24EI},w_C=w(\frac{l}{2})=\frac{5ql^4}{384EI}$$

5-2　简支梁承受荷载如图所示，试用积分法求θ_A，θ_B，并求$w_{\max}$所在截面的位置及该挠度的算式。



解：首先求支座反力F_A

$$F_A=\frac{\frac{1}{2}q_0 l\cdot\frac{l}{3}}{l}=\frac{1}{6}q_0 l$$



梁的弯矩方程

$$M(x)=F_A x-\frac{1}{2}q(x)\cdot x\cdot\frac{x}{3}=\frac{1}{6}q_0 lx-\frac{1}{6}\cdot\frac{q_0 l}{l}x^2$$

$$=\frac{1}{6}q_0(lx-\frac{x^3}{l}) \qquad (1)$$

则挠曲线的近似微分方程

$$EIw''=-M(x)=-\frac{1}{6}q_0(lx-\frac{x^3}{l}) \qquad (2)$$

积分两次，即得

$$EIw'=-\frac{1}{6}q_0(\frac{l}{2}x^2-\frac{x^4}{4l})+C \qquad (3)$$

$$EIw=-\frac{1}{6}q_0(\frac{l}{6}x^3-\frac{x^5}{20l})+Cx+D \qquad (4)$$

边界条件是两铰支端的挠度为零，即

$$x=0\text{时}\ y=0;x=l\text{时，}\ y=0$$

将其代入式（4），可得

$$C=\frac{7q_0 l^3}{360}，\ D=0$$

将积分常数C，D之值代入式（3），（4），梁的转角方程式和挠曲线方程为

$$\theta = w' = \frac{q_0}{EI}(-\frac{lx^2}{12} + \frac{x^4}{24l} + \frac{7l^3}{360}) \qquad (5)$$

$$w = \frac{q_0}{EI}(-\frac{lx^3}{36} + \frac{x^5}{120l} + \frac{7l^3 x}{360}) \qquad (6)$$

显然转角θ_A，θ_B分别是

$$\theta_A = \theta\Big|_{x=0} = \frac{q_0}{EI} \cdot \frac{7l^3}{360} = \frac{7q_0 l^3}{360EI} \text{（顺）}$$

$$\theta_B = \theta\Big|_{x=l} = \frac{q_0}{EI}(-\frac{l^3}{12} + \frac{l^3}{24} + \frac{7l^3}{360}) = -\frac{q_0 l^3}{45EI} \text{（逆）}$$

欲求$w_{\max}$的位置，首先令$w' = 0$，即有

$$15x^4 - 30l^2x^2 + 7l^4 = 0$$

解得

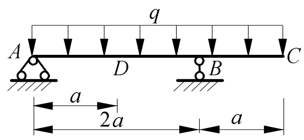
$$x = 0.52l$$

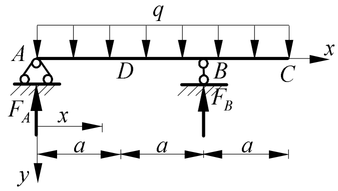
将$x = 0.52l$代入挠曲线方程，得梁的最大挠度计算式

$$w_{\max} = w\Big|_{x=0.52l} = \frac{q_0}{EI}[-\frac{l(0.52l)^3}{36} + \frac{(0.52l)^5}{120l} + \frac{7l^3 \times 0.52l}{360}] = 0.006\ 51\frac{q_0 l^4}{EI} \text{（向下）}$$

5-3　外伸梁承受均布荷载如图所示，试用积分法求θ_A，θ_B及w_D，w_C。

解：首先求支座反力为





$$F_A = \frac{2qa^2 - \frac{1}{2}qa^2}{2a} = \frac{3}{4}qa$$

$$F_B = \frac{\frac{1}{2}q(3a)^2}{2a} = \frac{9}{4}qa$$

对于第Ⅰ，第Ⅱ段梁的弯矩方程分别是

$$M_1(x) = \frac{3}{4}qax - \frac{1}{2}qx^2 \qquad (0 \le x \le 2a) \qquad (1)$$

$$M_2(x) = \frac{3}{4}qax + \frac{9qa}{4}(x - 2a) - \frac{1}{2}qx^2 \qquad (2a \le x \le 3a) \qquad (1')$$

令第Ⅰ段梁的挠曲线方程是w_1，于是得其挠曲线的近似微分方程式

$$EIw_1'' = -M_1(x) = -\frac{3}{4}qax + \frac{1}{2}qx^2 \qquad (0 \le x \le 2a) \qquad (2)$$

积分得

$$EIw_1' = -\frac{3}{8}qax^2 + \frac{1}{6}qx^3 + C_1 \qquad (3)$$

$$EIw_1 = -\frac{1}{8}qax^3 + \frac{1}{24}qx^4 + C_1x + D_1 \qquad (4)$$

令第Ⅱ段梁的挠曲线方程是w_2，得其挠曲线近似微分方程式

$$EIw_2'' = -M_2(x) = -\frac{3}{4}qax - \frac{9}{4}qa(x - 2a) + \frac{1}{2}qx^2 \ (2a \le x \le 3a) \qquad (2')$$

积分得

$$EIw_2' = -\frac{3}{8}qax^2 - \frac{9}{8}qa(x - 2a)^2 + \frac{1}{6}qx^3 + C_2 \qquad (3')$$

$$EIw_2 = -\frac{1}{8}qax^3 - \frac{3}{8}qa(x - 2a)^3 + \frac{1}{24}qx^4 + C_2x + D_2 \qquad (4')$$

利用点 B 挠曲线的连续条件

$$当x = 2a时, w_1' = w_2'与w_1 = w_2$$

于是由（3），（4），（3′），（4′）诸式得

$$C_1 = C_2, D_1 = D_2$$

利用边界条件

$$当x = 0时, w_1 = 0; x = 2a时, w_1 = w_2 = 0$$

由式（3）得 $EIw_1\Big|_{x=0} = 0 = D_1$

$$EIw_1\Big|_{x=2a} = 0 = -\frac{qa}{8}(2a)^3 + \frac{q}{24}(2a)^4 + C_1 \cdot 2a$$

即 $$C_1 = C_2 = \frac{qa^3}{6}, D_1 = D_2 = 0$$

将积分常数值分别代入式（3），（4），（3′），（4′）得到梁的第Ⅰ，第Ⅱ段的转角方程和挠曲线方程

第Ⅰ段 $(0 \le x \le 2a)$ $$\theta_1 = w_1' = \frac{1}{EI}(-\frac{3}{8}qax^2 + \frac{1}{6}qx^3 + \frac{qa^3}{6}) \tag{5}$$

$$w_1 = \frac{1}{EI}(-\frac{qa}{8}x^3 + \frac{q}{24}x^4 + \frac{qa^3}{6}x) \tag{6}$$

第Ⅱ段 $(2a \le x \le 3a)$

$$\theta_2 = w_2' = \frac{1}{EI}[-\frac{3}{8}qax^2 - \frac{9}{8}qa(x-2a)^2 + \frac{1}{6}qx^3 + \frac{1}{6}qa^3] \tag{5′}$$

$$w_2 = \frac{1}{EI}[-\frac{1}{8}qax^3 - \frac{3}{8}qa(x-2a)^3 + \frac{1}{24}qx^4 + \frac{1}{6}qa^3x] \tag{6′}$$

进而求得

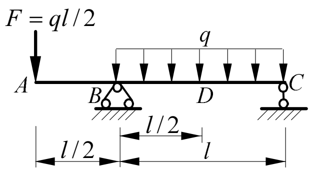
$$\theta_A = \theta_1\Big|_{x=0} = \frac{qa^3}{6EI} \text{（顺）}$$

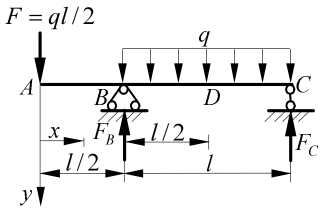
$$\theta_B = \theta_1\Big|_{x=2a} = \frac{1}{EI}[-\frac{3}{8}qa(2a)^2 + \frac{1}{6}q(2a)^3 + \frac{qa^3}{6}] = 0 \text{（逆）}$$

$$w_D = w_1\Big|_{x=a} = \frac{1}{EI}(-\frac{qa}{q} \cdot a^3 + \frac{q \cdot a^4}{24} + \frac{qa^3}{6}a) = \frac{qa^4}{12EI} \text{（向下）}$$

$$w_C = w_2\Big|_{x=3a} = \frac{1}{EI}[-\frac{qa}{q} \cdot (3a)^3 - \frac{3qa}{8}(3a-2a)^3 + \frac{q}{24} \cdot (3a)^4 + \frac{qa^3}{6} \cdot 3a] = \frac{qa^4}{8EI} \text{（向下）}$$

5-4　试用积分法求图示外伸梁的θ_A,θ_B及w_A,w_D。





解：首先求支反力

$$\sum M_C=0$$

$$F_B=\frac{\frac{1}{2}ql^2+F\cdot\frac{3}{2}l}{l}=\frac{1}{2}ql+\frac{ql}{2}\cdot\frac{3}{2}=\frac{5}{4}ql\ （\uparrow）$$

$$\sum F_y=0$$

$$F_C=F+ql-F_B=\frac{1}{2}ql+ql-\frac{5}{4}ql=\frac{1}{4}ql\ （\uparrow）$$

第Ⅰ段（AB段），第Ⅱ段（BC段）梁的弯矩方程分别是

$$M_1(x)=-\frac{1}{2}qlx\qquad(0\le x\le l/2)\qquad（1）$$

$$M_2(x)=-\frac{1}{2}qlx+\frac{5}{4}ql(x-\frac{l}{2})-\frac{1}{2}q(x-\frac{l}{2})^2\qquad(\frac{l}{2}\le x\le\frac{3}{2}l)\qquad（1'）$$

相应得挠曲线近似微分方程

$$EIw_1''=-M_1(x)=\frac{1}{2}qlx\qquad(0\le x\le\frac{l}{2})\qquad（2）$$

$$EIw_2''=-M_2(x)=\frac{1}{2}qlx-\frac{5}{4}ql(x-\frac{l}{2})+\frac{1}{2}q(x-\frac{l}{2})^2\qquad(\frac{l}{2}\le x\le\frac{3l}{2})\qquad（2'）$$

分别积分　$EIw_1'=\frac{1}{4}qlx^2+C_1$　（3）

$$EIw_1=\frac{1}{12}qlx^3+C_1x+D_1\qquad（4）$$

$$EIw'\ =\frac{1}{4}qlx\ -\frac{5}{8}ql(x-\frac{l}{2})\ +\frac{1}{6}q(x-\frac{l}{2})^3+C\qquad（3'）$$

$$EIw_2=\frac{1}{12}qlx^3-\frac{5}{24}ql(x-\frac{l}{2})^3+\frac{1}{24}q(x-\frac{l}{2})^4+C_2x+D_2\qquad（4'）$$

利用点B处梁的连续条件，即$x=\frac{l}{2}$时，有$w_1'=w_2'$，$w_1=w_2$而得到

$$C_1=C_2,\quad D_1=D_2$$

利用边界条件$x=\frac{l}{2}$时，$w_1=0$；$x=\frac{3l}{2}$时，$w_2=0$

即　$$EIw_2\Big|_{x=\frac{l}{2}}=0=\frac{1}{12}ql(\frac{l}{2})^3+C_1\cdot\frac{l}{2}+D_1=\frac{ql^4}{96}+\frac{l}{2}C_1+D_1\qquad（5）$$

$$EIw_2\Big|_{x=\frac{3l}{2}}=0=\frac{1}{12}ql(\frac{3l}{2})^3-\frac{5}{24}ql(\frac{3l}{2}-\frac{l}{2})^3+\frac{1}{24}q(\frac{3l}{2}-\frac{l}{2})^4+C_2\frac{3l}{2}+D_2$$

$$=\frac{11ql^4}{96}+\frac{3l}{2}C_2+D_2\qquad（6）$$

式（5）、（6）联解得　$C_1=-\frac{5ql^3}{48},D_1=\frac{ql^4}{24}$　（7）

将积分常数代入式（3）、（4）、（3′）、（4′），得到转角方程与挠曲线方程

$$\theta_1=w_1'=\frac{1}{EI}(\frac{ql}{4}x^2-\frac{5ql^3}{48})\qquad（0\le x\le\frac{l}{2}）\qquad（8）$$

$$w_1=\frac{1}{EI}(\frac{1}{12}qlx^3-\frac{5}{48}ql^3x+\frac{ql^4}{24})\quad（0\le x\le\frac{l}{2}）\tag{9}$$

$$\theta_2=w_2'=\frac{1}{EI}[\frac{1}{4}qlx^2-\frac{5}{8}ql(x-\frac{l}{2})^2+\frac{1}{6}q(x-\frac{l}{2})^3-\frac{5}{48}ql^3]\ （\frac{l}{2}\le x\le\frac{3l}{2}）\tag{8′}$$

$$w_2=\frac{1}{EI}[\frac{1}{12}qlx^3-\frac{5}{24}ql(x-\frac{l}{2})^3+\frac{1}{24}q(x-\frac{l}{2})^4-\frac{5}{48}ql^3x+\frac{ql^4}{24}]\ （\frac{l}{2}\le x\le\frac{3l}{2}）\tag{9′}$$

对所求特定点的转角或挠度，只须将其 x 坐标值，代入对应方程得

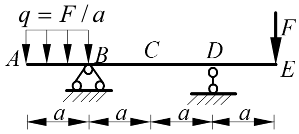
$$\theta_A=\theta_1\Big|_{x=0}=\frac{1}{EI}(-\frac{5ql^3}{48})=-\frac{5ql^3}{48EI}\quad（逆）$$

$$\theta_B=\theta_1\Big|_{x=\frac{l}{2}}=\frac{1}{EI}[\frac{ql}{4}(\frac{l}{2})^2-\frac{5ql^3}{48}]=-\frac{ql^3}{24EI}\quad（逆）$$

$$w_A=w_1\Big|_{x=0}=\frac{ql^4}{24EI}\quad(向下)$$

$$w_D=w_2\Big|_{x=l}=\frac{1}{EI}[\frac{1}{12}ql^4-\frac{5}{24}ql(l-\frac{l}{2})^3+\frac{1}{24}q(l-\frac{l}{2})^4-\frac{5}{48}ql^3\cdot l+\frac{ql^4}{24}]=-\frac{ql^4}{384EI}$$

5-5 外伸梁如图所示，试用积分法求 w_A,w_C和w_E。

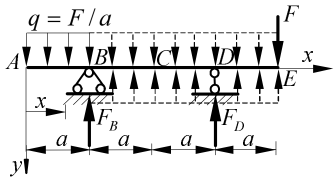


解：约束力

$$F_B=\frac{-Fa+qa\cdot\frac{5}{2}a}{2a}=\frac{3F}{4}$$

$$F_D=F+qa-F_B=\frac{5F}{4}$$

为了运算上的简化，在梁的 BE 段添加相等相反的均布荷载 $q=\dfrac{F}{a}$。(图中用虚线表示)



AB 段的挠曲线近似微分方程

$$EIw_1''=-M_1(x)=\frac{1}{2}qx^2\quad(0\le x\le a)\tag{1}$$

积分
$$EIw_1'=\frac{1}{6}qx^3+C_1\tag{2}$$

$$EIw_1=\frac{1}{24}qx^4+C_1x+D_1\tag{3}$$

BD 段的挠曲线近似微分方程

$$EIw_2''=-M_2(x)=\frac{1}{2}qx^2-F_B(x-a)-\frac{1}{2}q(x-a)^2$$

$$=\frac{1}{2}qx^2-\frac{3}{4}F(x-a)-\frac{1}{2}q(x-a)^2\qquad(a\le x\le 3a)\tag{1′}$$

积分
$$EIw_2'=\frac{1}{6}qx^3-\frac{3}{8}F(x-a)^2-\frac{1}{6}q(x-a)^3+C_2\tag{2′}$$

$$EIw_2=\frac{1}{24}qx^4-\frac{1}{8}F(x-a)^3-\frac{1}{24}q(x-a)^4+C_2x+D_2\tag{3′}$$

DE 段的挠曲线近似微分方程

$$EIw_3'' = -M_3(x) = \frac{1}{2}qx^2 - \frac{3}{4}F(x-a) - \frac{5}{4}F(x-3a) - \frac{1}{2}q(x-a)^2 \quad (3a \le x \le 4a) \ (1'')$$

积分 $$EIw_3' = \frac{1}{6}qx^3 - \frac{3}{8}F(x-a)^2 - \frac{5}{8}F(x-3a)^2 - \frac{1}{6}q(x-a)^3 + C_3 \quad (2'')$$

$$EIw_3 = \frac{1}{24}qx^4 - \frac{1}{8}F(x-a)^3 - \frac{5}{24}F(x-3a)^3 - \frac{1}{24}q(x-a)^4 + C_3x + D_3 \quad (3'')$$

利用梁在点 B 的连续条件，即 $x=a$ 时，$w_1' = w_2'$；$w_1 = w_2$，由式（2)，(3）和 $(2'')$，$(3'')$ 分别相等，得 $C_1 = C_2$，$D_1 = D_2$

同理利用点 D 的连续条件，得 $C_2 = C_3$，$D_2 = D_3$

于是有 $C_1 = C_2 = C_3$，$D_1 = D_2 = D_3$ （4）

利用边界条件，当 $x=a$ 时，$w_1 = w_2$=0；当 $x=3a$ 时，$w_2 = w_3$=0 有

$$EIw_1\Big|_{x=a} = \frac{qa^4}{24} + C_1a + D_1 = 0 \quad (5)$$

$$EIw_2\Big|_{x=3a} = \frac{81qa^4}{24} - \frac{1}{8}F(3a-a)^3 - \frac{1}{24}q(3a-a)^4 + 3C_2a + D_2 = 0 \quad (6)$$

由式（4)、(5)、(6）联立求解，并将 $q = \dfrac{F}{a}$ 代入得

$$C_1 = C_2 = C_3 = -\frac{5}{6}Fa^2,\quad D_1 = D_2 = D_3 = \frac{19}{24}Fa^3 \quad (7)$$

将式（7）代入式（2)、(3)、$(2')$、$(3')$、$(2'')$、$(3'')$，并将 $q = \dfrac{F}{a}$ 代入，得梁的位移方程

AB 段 $(0 \le x \le a)$

$$\theta_1 = w_1' = \frac{1}{EI}\left(\frac{Fx^3}{6a} - \frac{5}{6}Fa^2\right) \quad (8)$$

$$w_1 = \frac{1}{EI}\left(\frac{Fx^4}{24a} - \frac{5Fa^2}{6}x + \frac{19Fa^3}{24}\right) \quad (9)$$

BD 段 $(a \le x \le 3a)$

$$\theta_2 = w_2' = \frac{1}{EI}\left[\frac{Fx^3}{6a} - \frac{3F(x-a)^2}{8} - \frac{F(x-a)^3}{6a} - \frac{5Fa^2}{6}\right] \quad (8')$$

$$w_2 = \frac{1}{EI}\left[\frac{Fx^4}{24a} - \frac{F(x-a)^3}{8} - \frac{F(x-a)^4}{24a} - \frac{5Fa^2x}{6} + \frac{19Fa^3}{24}\right] \quad (9')$$

DE 段 $(3a \le x \le 4a)$

$$\theta_3 = w_3' = \frac{1}{EI}\left[\frac{Fx^3}{6a} - \frac{3F(x-a)^2}{8} - \frac{5F(x-3a)^2}{8} - \frac{F(x-a)^3}{6a} - \frac{5Fa^2}{6}\right] \quad (8'')$$

$$w_3 = \frac{1}{EI}\left[\frac{Fx^4}{24a} - \frac{F(x-a)^3}{8} - \frac{5F(x-3a)^3}{24} - \frac{F(x-a)^4}{24a} - \frac{5Fa^2x}{6} + \frac{19Fa^3}{24}\right] \quad (9'')$$

所求位移 w_A, w_C 和 w_E 是

$$w_A = w_1\Big|_{x=0} = \frac{19Fa^3}{24EI} \text{（向下）}$$

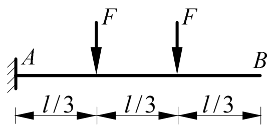
$$w_C = w_2\Big|_{x=2a} = \frac{1}{EI}[\frac{F(2a)^4}{24a} - \frac{F(2a-a)^3}{8} - \frac{F(2a-a)^4}{24a} - \frac{5Fa^2\cdot 2a}{6} + \frac{19Fa^3}{24}]$$

$$= -\frac{3Fa^3}{8EI}$$

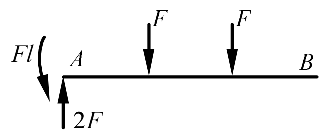
$$w_E = w_3\Big|_{x=4a} = \frac{1}{EI}[\frac{F(4a)^4}{24a} - \frac{F(4a-a)^3}{8} - \frac{5F(4a-3a)^3}{24}$$

$$-\frac{F(4a-a)^4}{24a} - \frac{5Fa^2\cdot 4a}{6} + \frac{19Fa^3}{24}] = \frac{7Fa^3}{6EI}$$ （向下）

5-6　试用积分法求图示悬臂梁 B 端的挠度 w_B。



解：本解借用奇异函数表示。由图 5-6a



$$M = -Fl + 2Fx - F < x - \frac{l}{3} > - F < x - \frac{2l}{3} >$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = -\frac{M}{EI}$$

$$w = -\frac{1}{EI}[-Fl\frac{< x-0 >^2}{2} + 2F\frac{< x-0 >^3}{6}$$

$$-F\frac{< x-\frac{l}{3} >^3}{6} - F\frac{< x-\frac{2l}{3} >^3}{6}] + Cx + D$$

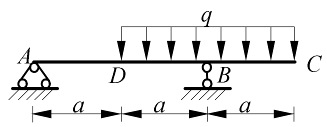
$$w(0)=0, D=0; \theta(0)=0, C=0$$

$$w = -\frac{1}{EI}[-\frac{Fl}{2}x^2 + \frac{F}{3}x^3 - \frac{F}{6}< x-\frac{l}{3} >^3 - \frac{F}{6}< x-\frac{2l}{3} >^3]$$

$$w = \frac{F}{6EI}[3lx^2 - 2x^3 + < x-\frac{l}{3} >^3 + < x-\frac{2l}{3} >^3]$$

$$w_B = w(l) = \frac{F}{6EI}(3l^3 - 2l^3 + \frac{8}{27}l^3 + \frac{l^3}{27}) = \frac{2Fl^3}{9EI}$$

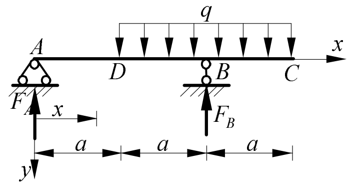
5-7　试用积分法求图示外伸梁的 θ_A 和 w_C。



解：根据静力平衡求支反力

$$F_A = \frac{\frac{1}{2}qa^2 - \frac{1}{2}qa^2}{2a} = 0$$

$$F_B = 2qa - F_A = 2qa$$



AD 段的挠曲线近似微分方程

$$EIw_1'' = -M_1(x) = 0 \qquad (0 \le x \le a) \qquad （1）$$

积分 $$EIw_1' = C_1 \qquad （2）$$

$$EIw_1 = C_1x + D_1 \qquad （3）$$

DB 段的挠曲线近似微分方程

$$EIw_2'' = -M_2(x) = \frac{1}{2}q(x-a)^2 \qquad (a \le x \le 2a) \qquad (1')$$

积分 $$EIw_2' = \frac{1}{6}q(x-a)^3 + C_2 \qquad (2')$$

$$EIw_2 = \frac{1}{24}q(x-a)^4 + C_2x + D_2 \tag{3'}$$

BC 段的挠曲线近似微分方程

$$EIw_3'' = -M_3(x) = \frac{1}{2}q(x-a)^2 - 2qa(x-2a) \qquad (2a \le x \le 3a) \tag{1''}$$

积分 $$EIw_3' = \frac{1}{6}q(x-a)^3 - qa(x-2a)^2 + C_3 \tag{2''}$$

$$EIw_3 = \frac{1}{24}q(x-a)^4 - \frac{qa}{3}(x-2a)^3 + C_3x + D_3 \tag{3''}$$

利用挠曲线在截面 D 和截面 B 的连续条件，得

$$C_1 = C_2 = C_3，\quad D_1 = D_2 = D_3 \tag{4}$$

利用边界条件，当 $x=0$ 时，$w_1 = 0$；当 $x = 2a$ 时，$w_2 = 0$，得

$$EIw_1\big|_{x=0} = D_1 = 0$$

$$EIy_2\bigg|_{x=2a} = \frac{q(2a-a)^4}{24} + C_2 \cdot 2a = 0，即：C_2 = -\frac{qa^3}{48}$$

由式（4）得 $C_1 = C_2 = C_3 = -\dfrac{qa^3}{48}$， $D_1 = D_2 = D_3 = 0$ （5）

将式（5）分别代入（2）、（3）、$(2')$、$(3')$、$(2'')$、$(3'')$ 得梁的位移方程

AD 段 $(0 \le x \le a)$

$$\theta_1 = w_1' = -\frac{qa^3}{48EI} \tag{6}$$

$$w_1 = -\frac{qa^3x}{48EI} \tag{7}$$

DB 段 $(a \le x \le 2a)$

$$\theta_2 = w_2' = \frac{1}{EI}[\frac{q(x-a)^3}{6} - \frac{qa^3}{48}] \tag{6'}$$

$$w_2 = \frac{1}{EI}[\frac{q(x-a)^4}{24} - \frac{qa^3x}{48}] \tag{7'}$$

BC 段 $(2a \le x \le 3a)$

$$\theta_3 = w_3' = \frac{1}{EI}[\frac{q(x-a)^3}{6} - qa(x-2a)^2 - \frac{qa^3}{48}] \tag{6''}$$

$$w_3 = \frac{1}{EI}[\frac{q(x-a)^4}{24} - \frac{qa(x-2a)^3}{3} - \frac{qa^3x}{48}] \tag{7''}$$

分别将 $x = 0$ 与 $x = 3a$ 代入上列有关式子，则求得 θ_A 和 w_C 为

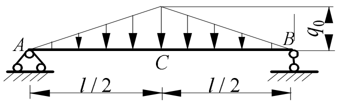
$$\theta_A = \theta_1\bigg|_{x=0} = -\frac{qa^3}{48EI} \quad （逆）$$

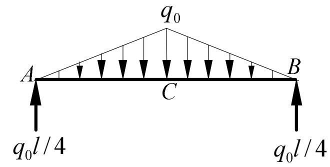
$$w_C = w_3\bigg|_{x=3a} = \frac{1}{EI}[\frac{q(3a-a)^4}{24} - \frac{qa(3a-2a)^3}{3} - \frac{qa^3 \cdot 3a}{48}] = -\frac{13qa^4}{48EI} \quad （向下）$$

5-8 简支梁承受荷载如图所示，试用积分法求 θ_A, θ_B 和 $w_{\max}$。

解：方法 1 由对称性知 $\theta_A = -\theta_B, w_{\max} = w(\frac{l}{2}), \theta_C = 0$

因此只计算左半段 $q(x)=\dfrac{2q_0}{l}x \quad (0\le x\le \dfrac{l}{2})$



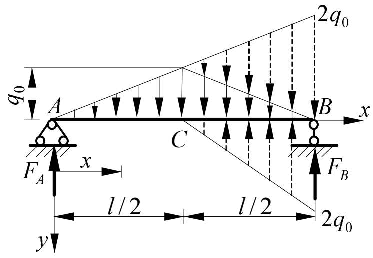


$$M=\frac{q_0 l}{4}x-\frac{1}{2}q(x)\cdot x\cdot\frac{x}{3}$$

$$M=\frac{q_0 l}{4}x-\frac{q_0}{3l}x^3$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}=-\frac{M}{EI}$$

$$EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}=-\frac{q_0 l}{4}x+\frac{q_0}{3l}x^3$$



$$EI\frac{\mathrm{d}w}{\mathrm{d}x}=-\frac{q_0 l}{8}x^2+\frac{q_0}{12l}x^4+C$$

$$\theta(\frac{l}{2})=0,-\frac{q_0 l}{8}(\frac{l}{2})^2+\frac{q_0}{12l}(\frac{l}{2})^4+C=0$$

$$C=\frac{5q_0 l^3}{192}$$

$$EIw=-\frac{q_0 l}{24}x^3+\frac{q_0}{60l}x^5+\frac{5q_0 l^3}{192}x$$

$$w_{\max}=w(\frac{l}{2})=\frac{1}{EI}[-\frac{q_0 l}{24}(\frac{l}{2})^3+\frac{q_0}{60l}(\frac{l}{2})^5+\frac{5q_0 l^3}{192}\frac{l}{2}]=\frac{q_0 l^4}{120EI}$$

$$\theta_A=-\theta_B=\frac{5q_0 l^3}{192EI}$$

方法 2 为了运算上方便，我们在梁的 CB 段添加相等相反的线分布荷载如图中虚线所示。

由于对称，约束力 $F_A=F_B=\dfrac{1}{2}q_0\cdot\dfrac{l}{2}=\dfrac{q_0 l}{4}$，梁 AC 段的挠曲线近似微分方程

$$EIw_1''=-\frac{q_0 lx}{4}+\frac{1}{2}\cdot\frac{x\cdot 2q_0}{l}\cdot x\cdot\frac{x}{3}=-\frac{q_0 lx}{4}+\frac{q_0 x^3}{3l} \quad (0\le x\le\frac{l}{2}) \tag{1}$$

积分 $$EIw_1'=-\frac{q_0 lx^2}{8}+\frac{q_0 x^4}{12l}+C_1 \tag{2}$$

$$EIw_1=-\frac{q_0 lx^3}{24}+\frac{q_0 x^5}{60l}+C_1 x+D_1 \tag{3}$$

梁 CB 段的挠曲线近似微分方程式

$$EIw_2''=-\frac{q_0 lx}{4}+\frac{q_0 x^3}{3l}-\frac{2q_0(x-\frac{l}{2})^3}{3l} \quad (\frac{l}{2}\le x\le l) \tag{1'}$$

积分 $$EIw_2'=-\frac{q_0 lx^2}{8}+\frac{q_0 x^4}{12l}-\frac{2q_0(x-\frac{l}{2})^4}{12l}+C_2 \tag{2'}$$

$$EIw_2=-\frac{q_0 lx^3}{24}+\frac{q_0 x^5}{60l}-\frac{2q_0(x-\frac{l}{2})^5}{60l}+C_2 x+D_2 \tag{3'}$$

利用梁在截面 C 处的连续条件，即 $x=\dfrac{l}{2}$ 时，$w_1'=w_2'$，$w_1=w_2$ 得

$$C_1=C_2，\ D_1=D_2 \tag{4}$$

边界条件是，当 $x=0$ 时，$w_1=0$；当 $x=l$ 时，$w_2=0$。利用边界条件有

$$EIw_1\Big|_{x=0}=D_1=0$$

$$EIw_2\,|_{x=l}=-\frac{q_0l^4}{24}+\frac{q_0l^5}{60l}-\frac{2q_0(l-\frac{l}{2})^5}{60l}+C_2l=0$$

即

$$C_2=\frac{5q_0l^3}{192} \tag{5}$$

将式（4）与（5）代入式（2）、（3）、(2′)、(3′)，得梁的位移方程：

AC 段 $(0\le x\le\dfrac{l}{2})$

$$\theta_1=w_1'=\frac{1}{EI}(-\frac{q_0lx^2}{8}+\frac{q_0x^4}{12l}+\frac{5q_0l^3}{192}) \tag{6}$$

$$w_1=\frac{1}{EI}(-\frac{q_0lx^3}{24}+\frac{q_0x^5}{60l}-\frac{5q_0l^3}{192}x) \tag{7}$$

CB 段 $(\dfrac{l}{2}\le x\le l)$

$$\theta_2=w_2'=\frac{1}{EI}(-\frac{q_0lx^2}{8}+\frac{q_0x^4}{12l}-\frac{2q_0(x-\frac{l}{2})^4}{12l}+\frac{5q_0l^3}{192}) \tag{6′}$$

$$w_2=\frac{1}{EI}(-\frac{q_0lx^3}{24}+\frac{q_0x^5}{60l}-\frac{2q_0(x-\frac{l}{2})^5}{60l}+\frac{5q_0l^3x}{192}) \tag{7′}$$

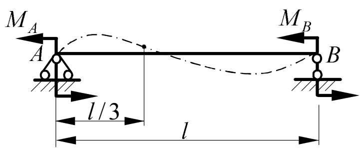
由于对称性，所以

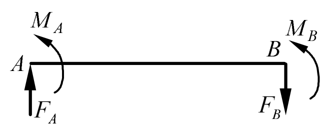
$$\theta_A=-\theta_B=\theta_1\Big|_{x=0}=\frac{5q_0l^3}{192EI}$$

也因为对称，最大挠度 $w_{\max}$ 必然发生在跨中截面，即

$$w_{\max}=w_1\Big|_{x=\frac{l}{2}}=\frac{1}{EI}[-\frac{q_0l(\frac{l}{2})^3}{24}+\frac{q_0(\frac{l}{2})^5}{60l}+\frac{5q_0l^3}{192}\cdot\frac{l}{2}]=\frac{q_0l^4}{120EI}$$

5-9　在简支梁的左、右支座上，分别有力偶 M_A和M_B 作用，如图所示。为使该梁挠曲线的拐点位于距左端 $\dfrac{l}{3}$ 处，试求 M_A与M_B 间的关系。





解：图 5-9a，$\sum M_B=0$

$$F_Al=M_A+M_B,F_A=\frac{M_A+M_B}{l}$$

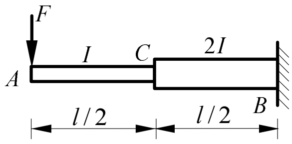
$$M=F_Ax-M_A=\frac{M_A+M_B}{l}x-M_A$$

$$\frac{\mathrm{d}^2w}{\mathrm{d}x^2}=-\frac{M}{EI}\Big|_{x=\frac{l}{3}}=0$$

$$x=\frac{l}{3}\text{时},M=\frac{M_A+M_B}{l}\cdot\frac{l}{3}-M_A=0$$

$$\frac{M_B}{3}-\frac{2M_A}{3}=0,M_B=2M_A$$

5-10　变截面悬臂梁及其荷载如图所示，试用积分法求梁 A 端的挠度 w_A。



解：（1）AC 段 $M=-Fx$

$$EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}=Fx \qquad (0\le x\le\frac{l}{2})$$

$$EI\frac{\mathrm{d}w_1}{\mathrm{d}x}=\frac{1}{2}Fx^2+C_1 \qquad (0\ x\le\frac{l}{2})$$

$$EIw_1=\frac{1}{6}Fx^3+C_1x+D_1$$

（2）CB 段 $M=-Fx$

$$2EI\frac{\mathrm{d}^2 w_2}{\mathrm{d}x^2}=Fx \qquad (\frac{l}{2}\le x\le l)$$

$$2EI\frac{\mathrm{d}w_2}{\mathrm{d}x}=\frac{1}{2}Fx^2+C_2$$

$$2EIw_2=\frac{F}{6}x^3+C_2x+D_2$$

由 $\theta_B=0$，得 $\frac{1}{2}Fl^2+C_2=0,C_2=-\frac{F}{2}l^2$

由 $w_B=0$，得 $\frac{F}{6}l^3+(-\frac{F}{2}l^2)\cdot l+D_2=0,D_2=\frac{1}{3}Fl^3$

$$2EI\theta_2=\frac{F}{2}x^2-\frac{F}{2}l^2$$

$$2EIw_2=\frac{F}{6}x^3-\frac{F}{2}l^2x+\frac{1}{3}Fl^3$$

由 $\theta_1(\frac{l}{2})=\theta_2(\frac{l}{2})$，$\frac{1}{EI}[\frac{F}{2}\cdot(\frac{l}{2})^2+C_1]=\frac{1}{2EI}[\frac{F}{2}(\frac{l}{2})^2-\frac{F}{2}l^2]$

$$C_1=\frac{-5}{16}Fl^2$$

由 $w_1(\frac{l}{2})=w_2(\frac{l}{2})$，$\frac{1}{EI}[\frac{F}{6}\cdot(\frac{l}{2})^3+(-\frac{5}{16}Fl^2)\frac{l}{2}+D_1]$

$$=\frac{1}{2EI}[\frac{F}{6}(\frac{l}{2})^3-\frac{F}{2}l^2\cdot(\frac{l}{2})+\frac{1}{3}Fl^3]$$

$$D_1=\frac{3}{16}Fl^3$$

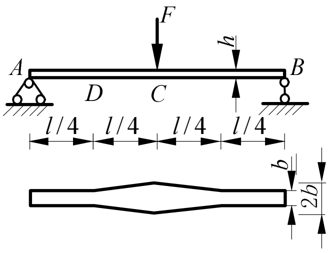
$$w_1=\frac{1}{EI}[\frac{F}{6}x^3+(-\frac{11}{48})Fl^2x+\frac{3}{16}Fl^3]$$

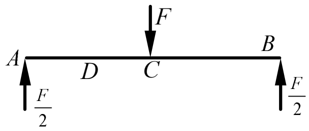
$$w_A=w(0)=\frac{3Fl^2}{16EI}$$

5-11　变截面简支梁及其荷载如图所示，试用积分法求跨中挠度 w_C。

解：（1）挠曲线方程

由于对称，考虑梁的左半部分 AC。





AD 段 $\qquad (0 \le x \le \dfrac{l}{4})$

$$I_1 = \frac{bh^3}{12}, M_1 = \frac{F}{2}x$$

$$\frac{\mathrm{d}^2 w_1}{\mathrm{d}x^2} = -\frac{M_1}{EI_1} = -\frac{Fx}{2EI_1}$$

$$\frac{\mathrm{d}w_1}{\mathrm{d}w} = \theta_1 = \frac{-F}{2EI_1}\cdot\frac{x^2}{2} + C_1$$

$$w_1 = \frac{-F}{2EI_1}\cdot\frac{x^3}{6} + C_1 x + D_1 \qquad （1）$$

由 $w_1(0) = 0$ 得，$D_1 = 0$

DC 段 $(\dfrac{l}{4} \le x \le \dfrac{l}{2})$

$$M_2 = \frac{F}{2}x，\ I_2 = \frac{\frac{4bx}{l}\cdot h^3}{12} = \frac{bh^3}{3l}x = \frac{4I_1}{l}x$$

$$\frac{\mathrm{d}^2 w_2}{\mathrm{d}x^2} = -\frac{M_2}{EI_2} = -\frac{Fl}{8EI_1}$$

$$\theta_2 = \frac{\mathrm{d}w_2}{\mathrm{d}x} = -\frac{Fl}{8EI_1}x + C_2$$

$$\theta_C = \theta_2(\frac{l}{2}) = 0, -\frac{Fl}{8EI_1}\cdot\frac{l}{2} + C_2 = 0, C_2 = \frac{Fl^2}{16EI_1} \qquad （2）$$

$$w_2 = -\frac{Fl}{16EI_1}x^2 + C_2 x + D_2 \qquad （3）$$

由 $x = \dfrac{l}{4}, \theta_1 = \theta_2$

$$\frac{-F}{2EI_1}\cdot\frac{1}{2}(\frac{l}{4})^2 + C_1 = -\frac{Fl}{8EI_1}\cdot\frac{l}{4} + C_2，\ C_1 - C_2 = -\frac{Fl^2}{64EI_1} \qquad （4）$$

式（2）代入（4）得 $\quad C_1 = \dfrac{3Fl^2}{64EI_1} \qquad （5）$

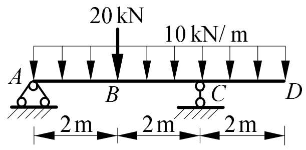
由 $w_1(\dfrac{l}{4}) = w_2(\dfrac{l}{4})$

$$\frac{-F}{2EI_1}\cdot\frac{1}{6}\cdot(\frac{l}{4})^3 + \frac{3Fl^2}{64EI_1}\cdot\frac{l}{4} = -\frac{Fl}{16EI_1}\cdot(\frac{l}{4})^2 + \frac{Fl^2}{16EI_1}\cdot\frac{l}{4} + D_2$$

$$D_2 = \frac{-Fl^3}{768EI_1}$$

$$w_C = w_2(\frac{l}{2}) = -\frac{Fl}{16EI_1}(\frac{l}{2})^2 + \frac{Fl^2}{16EI_1}\cdot\frac{l}{2} - \frac{Fl^3}{768EI_1} = \frac{11Fl^3}{768EI_1} = \frac{11Fl^3}{64Ebh^3}$$

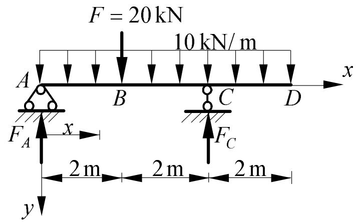
5-12　试用积分法求图示外伸梁 w_B 及 w_D 的值。已知梁由18 号工字钢制成，$E=210\,\text{GPa}$。



解：首先求支座反力 F_A, F_C

$$F_A=\frac{20\times2+10\times4\times2-10\times2\times1}{4}=25\,\text{kN}$$

$$F_C=\frac{20\times2+10\times6\times3}{4}=55\,\text{kN}$$



外伸梁的 AB 段的挠曲线近似微分方程

$$EIw_1''=-M_1(x)=-F_Ax+\frac{1}{2}qx^2 \qquad (1)$$

$$(0\le x\le a)$$

积分 $$EIw_1'=-\frac{F_Ax^2}{2}+\frac{qx^3}{6}+C_1 \qquad (2)$$

$$EIw_1=-\frac{F_Ax^3}{6}+\frac{qx^4}{24}+C_1x+D_1 \qquad (3)$$

BC 段的挠曲线近似微分方程

$$EIw_2''=-M_2(x)=-F_Ax+\frac{1}{2}qx^2+F(x-a) \qquad (a\le x\le 2a) \qquad (1')$$

积分 $$EIw_2'=-\frac{F_Ax^2}{2}+\frac{qx^3}{6}+\frac{F(x-a)^2}{2}+C_2 \qquad (2')$$

$$EIw_2=-\frac{F_Ax^2}{6}+\frac{qx^4}{24}+\frac{F(x-a)^3}{6}+C_2x+D_2 \qquad (3')$$

CD 段的挠曲线近似微分方程

$$EIw_3''=-F_Ax+\frac{qx^2}{2}+F(x-a)-F_B(x-2a) \qquad (2a\le x\le 3a) \qquad (1'')$$

积分 $$EIw_3'=-\frac{F_Ax^2}{2}+\frac{qx^3}{6}+\frac{F(x-a)^2}{2}-\frac{F_B(x-2a)^2}{2}+C_3 \qquad (2'')$$

$$EIw_3=-\frac{F_Ax^3}{6}+\frac{qx^4}{24}+\frac{F(x-a)^3}{6}-\frac{F_B(x-2a)^3}{6}+C_3x+D_3 \qquad (3'')$$

利用截面 B，C 的连续条件得

$$C_1=C_2=C_3\,,\quad D_1=D_2=D_3 \qquad (4)$$

边界条件是 $x=0$ 时，$w_1=0$；与 $x=2a$ 时，$y_2=y_3=0$

于是 $$EIw_1\big|_{x=0}=D_1=0$$

$$EIw_2\Big|_{x=2a}=-\frac{F_A(2a)^3}{6}+\frac{q(2a)^4}{24}+\frac{F(2a-a)^3}{6}+C_2\cdot 2a=0$$

即 $$C_2=\frac{2F_Aa^2}{3}-\frac{qa^3}{3}-\frac{Fa^2}{12} \qquad (5)$$

将式（5）、（4）分别代入式（2）、（3）、$(2')$、$(3')$、$(2'')$、$(3'')$，得外伸梁的转角方程与挠曲线方程

AB 段 $(0\le x\le 2\,\text{m})$

$$\theta_1=w_1'=\frac{1}{EI}[-\frac{F_Ax^2}{2}+\frac{qx^3}{6}+(\frac{2F_Aa^2}{3}-\frac{qa^3}{3}-\frac{Fa^2}{12})] \qquad (6)$$

$$w_1 = \frac{1}{EI}[-\frac{F_A x^3}{6} + \frac{qx^4}{24} + (\frac{2F_A a}{2} - \frac{qa^3}{3} - \frac{Fa^2}{12})x] \qquad (7)$$

BC 段$(2\ \mathrm{m} \le x \le 4\ \mathrm{m})$

$$\theta_2 = w_2' = \frac{1}{EI}[-\frac{F_A x^2}{2} + \frac{qx^3}{6} + \frac{F(x-a)^2}{2} + (\frac{2F_A a^2}{3} - \frac{qa^3}{3} - \frac{Fa^2}{12})] \qquad (6')$$

$$w_2 = \frac{1}{EI}[-\frac{F_A x^3}{6} + \frac{qx^4}{24} + \frac{F(x-a)^3}{6} + (\frac{2F_A a^2}{3} - \frac{qa^3}{3} - \frac{Fa^2}{12})x] \qquad (7')$$

CD 段$(4\ \mathrm{m} \le x \le 6\ \mathrm{m})$

$$\theta_3 = w_3' = \frac{1}{EI}[-\frac{F_A x^2}{2} + \frac{qx^3}{6} + \frac{F(x-a)^2}{2} - \frac{F_B(x-2a)^2}{2} + (\frac{2F_A a^2}{3} - \frac{qa^3}{3} - \frac{Fa^2}{12})] \qquad (6'')$$

$$w_3 = \frac{1}{EI}[-\frac{F_A x^3}{6} + \frac{qx^4}{24} + \frac{F(x-a)^3}{6} - \frac{F_B(x-2a)^3}{6} + (\frac{2F_A a}{3} - \frac{qa^3}{3} - \frac{Fa}{12})x] \qquad (7'')$$

由型钢表知 18 号工字钢的惯性矩为 $I = 1.66 \times 10^{-5}\ \mathrm{m}^4$

$x = 2\ \mathrm{m}$ 及有关数据代入式（7），即求得 w_B 为

$$w_B = w_1\Big|_{x=2} = \frac{1}{EI}[-\frac{25\times10^3\times2^3}{6} + \frac{10\times10^3\times2^4}{24} + (\frac{2\times25\times10^3\times2^2}{3} - \frac{10\times10^3\times2^3}{3} - \frac{20\times10^3\times2^2}{12})\times2]$$

$$= \frac{4\times10^4}{2.1\times10^{11}\times1.66\times10^{-5}} = 0.0115\ \mathrm{m} = 11.5\ \mathrm{mm} \quad （向下）$$

同理将 $x = 6\ \mathrm{m}$ 代入式$(7'')$求得 w_D

$$w_D = w_3\Big|_{x=6} = \frac{1}{2.1\times10^{11}\times1.66\times10^{-5}}[-\frac{25\times10^3\times6^3}{6} + \frac{10\times10^3\times6^4}{24} + \frac{20\times10^3\times(6-2)^3}{6} - \frac{55\times10^3\times(6-2\times2)^3}{6} + (\frac{2\times25\times10^3\times2^2}{3} - \frac{10\times10^3\times2^3}{3} - \frac{20\times10^3\times2^2}{12})\times6]$$

$$= -0.00575\ \mathrm{m} = -5.75\ \mathrm{mm}（向上）$$

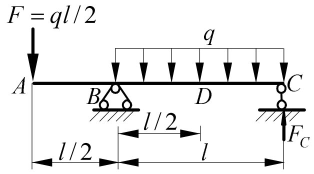
5-13　试按迭加原理并利用附录 IV 求解习题 5-4。

解： $w_{A1} = \dfrac{\frac{1}{2}ql(\frac{l}{2})^3}{3EI}$

$$\theta_{A1} = \frac{-\frac{1}{2}ql\cdot l^2}{2EI} = \frac{-ql^3}{4EI}$$

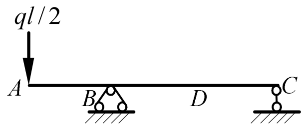
$$\theta_{B1} = \frac{-\frac{1}{4}ql^2\cdot l}{3EI} = \frac{-ql^3}{12EI}$$

$$w_{D1}=\frac{\frac{1}{4}ql^2\cdot l^2}{16EI}=\frac{ql^4}{64EI}$$



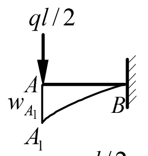
$$w_{A2}=\left|\theta_{B1}\right|\cdot\frac{l}{2}=\frac{ql^4}{24EI}$$

$$\theta_{B2}=\frac{ql^3}{24EI}$$



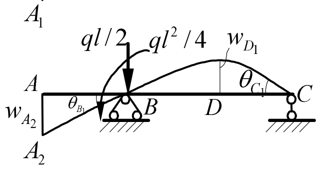
$$w_{A3}=-\theta_{B2}\cdot\frac{l}{2}=\frac{-ql^4}{48EI}$$

$$w_{D2}=\frac{5ql^4}{384EI}$$



$$w_A=w_{A1}+w_{A2}+w_{A3}$$

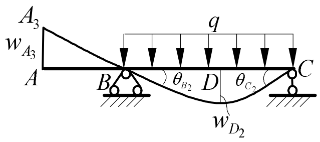
$$=\frac{ql^4}{48EI}+\frac{ql^4}{24EI}-\frac{ql^4}{48EI}=\frac{ql^4}{24EI}\quad（向下）$$



$$w\quad=w\quad+w\quad=\frac{-ql^4}{64EI}+\frac{5ql^4}{384EI}=-\frac{ql^4}{384EI}\quad（向上）$$

$$\theta_A=\theta_{A1}+\theta_{B1}+\theta_{B2}=-\frac{ql^3}{4EI}-\frac{ql^3}{12EI}+\frac{ql^3}{24EI}=-\frac{7ql^3}{24EI}$$

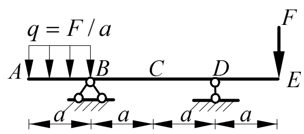
（逆）

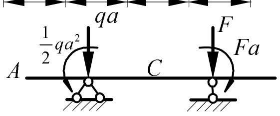


$$\theta_B=\theta_{B1}+\theta_{B2}=-\frac{ql^3}{12EI}+\frac{ql^3}{24EI}=-\frac{ql^3}{24EI}\quad（逆）$$

5-14　试按迭加原理并利用附录 IV 求解习题 5-5。

解: 分析梁的结构形式，而引起 BD 段变形的外力则如图（a）所示，即弯矩 $\frac{1}{2}qa^2$ 与弯矩 Fa。

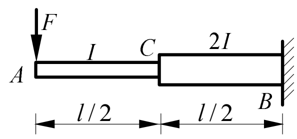




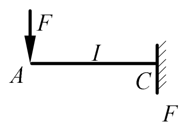
由附录（Ⅳ）知，跨长 l 的简支梁的梁一端受一集中力偶 M 作用时，跨中点挠度为 $w=\frac{Ml^2}{16EI}$。用到此处再利用迭加原理得截面 C 的挠度 w_C

$$w_C=\frac{\frac{1}{2}qa^2(2a)^2}{16EI}+\frac{Fa(2a)^2}{16EI}=\frac{\frac{F}{a}\cdot a^4}{8EI}+\frac{2Fa^3}{8EI}=\frac{3Fa^3}{8EI}\quad（向上）$$

5-15　试按迭加原理并利用附录 IV 求解习题 5-10。

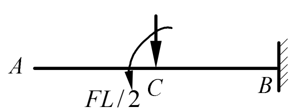


解:　$$w_{A1}=\frac{F\cdot(\frac{l}{2})^3}{3EI}=\frac{Fl^3}{24EI}$$



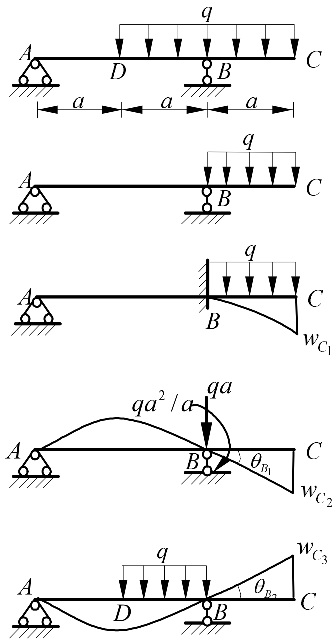
$$w_C=\frac{F(\frac{l}{2})^3}{3E(2I)}+\frac{\frac{Fl}{2}\cdot(\frac{l}{2})^2}{2E(2I)}=\frac{5Fl^3}{96EI}$$

$$\theta_C=\frac{\frac{Fl}{2}\cdot\frac{l}{2}}{E\cdot 2I}+\frac{F\cdot(\frac{l}{2})^2}{2E\cdot 2I}=\frac{3Fl^2}{16EI}$$



$$w_A = w_{A1} + w_C + \theta_C \cdot \frac{l}{2} = \frac{3Fl^3}{16EI}$$

5-16　试按迭加原理并利用附录 IV 求解习题 5-7 中的 w_C。



解：原梁可分解成图 5-16a 和图 5-16d 迭加，而图 5-16a 又可分解成图 5-16b 和 5-16c。

由附录Ⅳ得

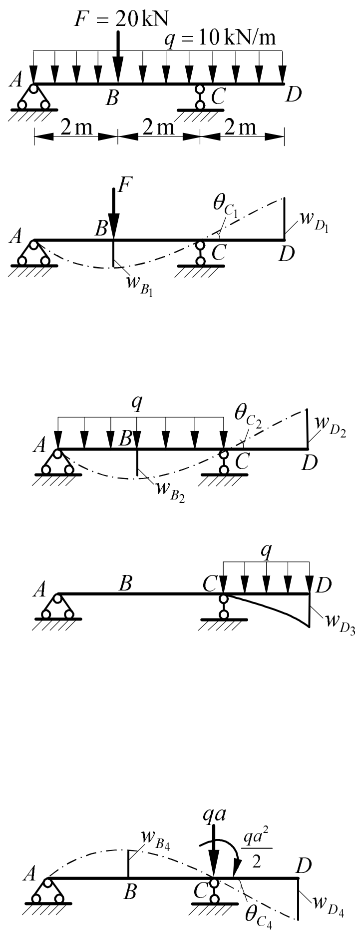
$$w_{C1} = \frac{qa^4}{8EI}$$

$$w_{C2} = \theta_{B1} \cdot a = \frac{\frac{1}{2}qa^2 \cdot 2a}{3EI} \cdot a = \frac{qa^4}{3EI}$$

$$w_{C3} = \theta_{B2} \cdot a = -\frac{qa^2 \cdot (3a)^2}{24EI(2a)} \cdot a = -\frac{9qa^4}{48EI}$$

$$w_C = w_{C1} + w_{C2} + w_{C3} = \frac{13qa^4}{48EI}$$

5-17　试按迭加原理并利用附录 IV 求解习题 5-12。



解：在集中荷载 F 的单独作用下，梁的变形如图（a）所示。由附录（IV）知

$$w_{B1} = \frac{F(2a)^3}{48EI} = \frac{Fa^3}{6EI}$$

$$w_{D1} = \theta_{C1} \cdot a = \frac{F(2a)^2}{16EI}a = \frac{Fa^3}{4EI}$$

在 AC 段上分布荷载作用下，梁的变形如图（b）所示。由附录（IV）得

$$w_{B2} = \frac{5q(2a)^4}{384EI} = \frac{5qa^4}{24EI}$$

$$w_{D2} = \theta_{C2} \cdot a = \frac{q(2a)^3}{24EI}a = \frac{qa^4}{3EI}$$

在 CD 段均布荷载作用下，梁的变形分做两部分考虑。先考虑 CD 段的变形引起的点 D 挠度 w_{D3} 即如图（c）所示，从附录（IV）中得

$$w_{D3} = \frac{qa^4}{8EI}$$

然后考虑 AC 段的变形，引起的点 D 挠度 w_{D4}，点 B 挠度 w_{B4}，如图（d）所示，引起 AC 段变形的仅是弯矩 $\frac{1}{2}qa^2$，根据附录（IV）得

$$w_{B4} = \frac{\frac{1}{2}qa^2(2a)^2}{16EI} = \frac{qa^4}{8EI}$$

$$w_{D4}=\theta_{C4}\cdot a=\frac{\frac{1}{2}qa^2\cdot 2a}{3EI}a=\frac{qa^4}{3EI}$$

由迭加原理得

$$w_B=w_{B1}+w_{B2}-w_{B4}=\frac{Fa^3}{6EI}+\frac{5qa^4}{24EI}-\frac{qa^4}{8EI}=\frac{Fa^3}{6EI}+\frac{qa^4}{12EI}=\frac{a^3}{12EI}(2F+qa)$$

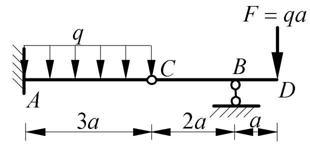
$$w_D=-w_{D1}-w_{D2}+w_{D3}+w_{D4}=-\frac{Fa^3}{4EI}-\frac{qa^4}{3EI}+\frac{qa^4}{8EI}+\frac{qa^4}{3EI}=-\frac{a^3}{8EI}(2F-qa)$$

由型钢表知 18 号工字钢的惯性矩 $I=1.66\times10^{-5}\ \mathrm{m}^4$，将数据代入，得外伸梁截面 B，截面 D 的挠度 w_B, w_D

$$w_B=\frac{2^3}{12\times2.1\times10^{11}\times1.66\times10^{-5}}(2\times20\times10^3+10\times10^3\times2)$$
$$=0.0115\ \mathrm{m}=11.5\ \mathrm{mm}\ （向下）$$

$$w_D=\frac{2^3}{8\times2.1\times10^{11}\times1.66\times10^{-5}}(2\times20\times10^3-10\times10^3\times2)$$
$$=-0.00575\ \mathrm{m}=-5.75\ \mathrm{mm}\ （向上）$$

5-18　试按迭加原理求图示梁中间铰 C 处的挠度 w_C，并描出梁挠曲线的大致形状。已知 EI 为常量。



解：（a）由图 5-18a-1

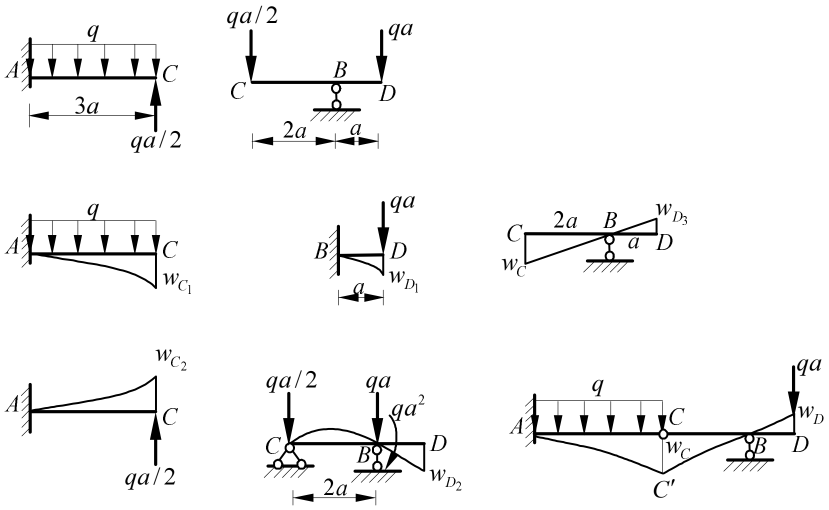
$$w_C=\frac{q(3a)^4}{8EI}-\frac{\frac{qa}{2}(3a)^3}{3EI}$$
$$=\frac{135qa^4}{24EI}$$

$$w_{D1}=\frac{qa\cdot a^3}{3EI}$$

$$w_{D2}=\frac{qa^2\cdot 2a}{3EI}\cdot a$$
$$=\frac{2qa^4}{3EI}$$

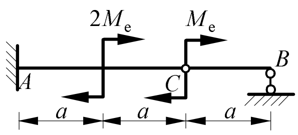
$$w_{D3}=-\frac{1}{2}w_c$$
$$=\frac{-135qa^4}{48EI}$$

$$w_D=(\frac{1}{3}+\frac{2}{3}+\frac{-135}{48})\frac{qa^4}{EI}$$
$$=-\frac{29qa^4}{16EI}$$



（b）由图 5-18b-1

$$F_B=F_C=\frac{M_e}{a}$$



$$w_{C1}=w_{D1}+\theta_{D1}\cdot a$$

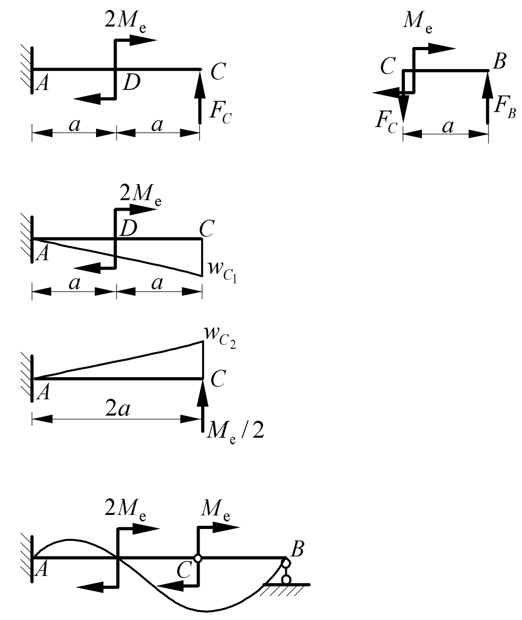
$$=\frac{2M_{\mathrm{e}}a^2}{2EI}+\frac{2M_{\mathrm{e}}a}{EI}\cdot a=\frac{3M_{\mathrm{e}}a^2}{EI}$$

$$w_{C2}=\frac{-\frac{M_{\mathrm{e}}}{a}a^3}{3EI}=-\frac{M_{\mathrm{e}}a^2}{3EI}$$

$$w_C=w_{C1}+w_{C2}=\frac{8M_{\mathrm{e}}a^2}{3EI}$$

$$w_D=w_{D1}+w_{D2}$$

$$=\frac{2M_{\mathrm{e}}a^2}{2EI}+\frac{-\frac{M_{\mathrm{e}}}{a}a^3}{3EI}+\frac{-M_{\mathrm{e}}a^2}{2EI}=\frac{M_{\mathrm{e}}a^2}{6EI}$$



5-19　试按迭加原理求图示平面折杆自由端截面 C 的铅垂位移和水平位移。已知杆各段的横截面面积均为 A，弯曲刚度均为 EI。

解：$\delta_{Cy1}=\dfrac{Fa^3}{3EI}$

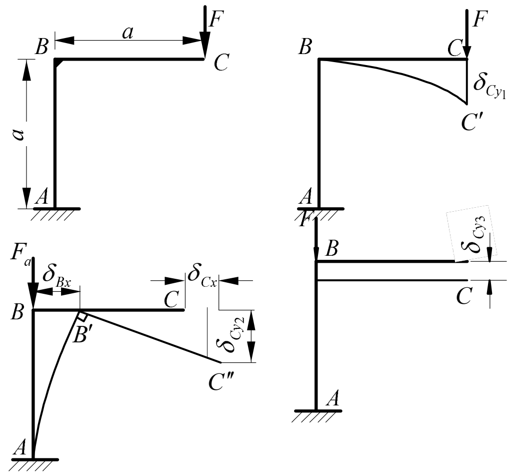
$$\delta_{Cy2}=\theta_B\cdot a=\frac{Fa\cdot a}{EI}\cdot a=\frac{Fa^3}{EI}$$

$$\delta_{Cx}=\delta_{Bx}=\frac{Fa\cdot a^2}{2EI}=\frac{Fa^3}{2EI}$$

$$\delta_{Cy3}=\frac{Fa}{EA}$$

$$\delta_{Cy}=\delta_{Cy1}+\delta_{Cy2}+\delta_{Cy3}$$

$$=\frac{4Fa^3}{3EI}+\frac{Fa}{EA}$$



5-20　图示结构中，在截面 A，D 处承受一对等值、反向的力 F，已知各段杆的 EI 均相等。试按叠加原理求 A，D 两截面间的相对位移。

解：（1）弯矩图如图 5-20a

（2）先将 BC 段刚化，图 5-20b，则

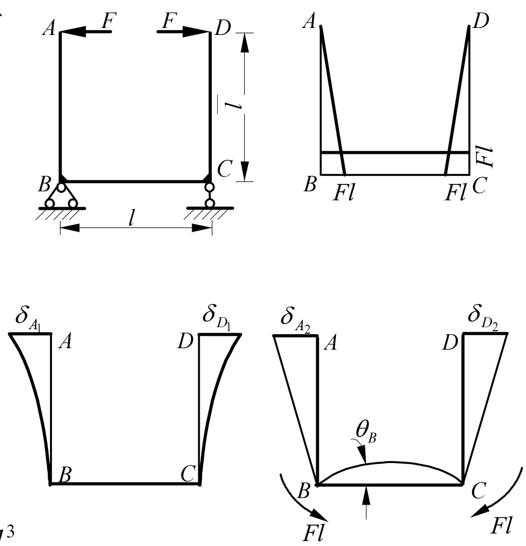
$$\delta_{A1}=\delta_{D1}=\frac{Fl^3}{3EI}$$

（3）BC 段弯曲在截面 B,C 产生转角使 A，D 发生牵连位移。

$$\theta_B=\frac{Fl\cdot l}{3EI}+\frac{Fl\cdot l}{6EI}=\frac{Fl^2}{2EI}$$

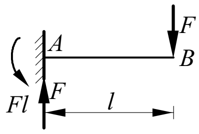
$$\delta_{A2}=\delta_{D2}=\theta_B\cdot l=\frac{Fl^3}{2EI}$$

$$\delta_{AD}=2(\delta_{A1}+\delta_{A2})=2(\frac{1}{3}+\frac{1}{2})\frac{Fl^3}{EI}=\frac{5Fl^3}{3EI}$$



*5-21　试用初参数法验算附录 IV 中第 2 项中梁的最大挠度及梁端转角的表达式。

解： 由公式（5-4），$q(x)=0,\theta_0=0,w_0=0,$则



$$EIw=-\frac{F_{S0}}{6}x^3-\frac{M_0}{2}x^2$$

$$EIw=-\frac{F}{6}x^3+\frac{Fl}{2}x^2$$

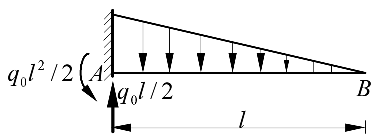
$$EI\theta=-\frac{F}{2}x^2+Flx$$

$$w_{\max}=\frac{1}{EI}(-\frac{F}{6}l^3+\frac{Fl}{2}\cdot l^2)=\frac{Fl^3}{3EI}$$

$$\theta_B=\frac{1}{EI}(-\frac{F}{2}l^2+Fl\cdot l)=\frac{Fl^2}{2EI}$$

*5-22　试用初参数法验算附录 IV 中第 5 项中梁的最大挠度及梁端转角的表达式。

解：　$q(x)=-(1-\frac{x}{l})q_0$



$$EIw=\iiiint q_0\ (1-\frac{x}{l})\,\mathrm{d}\,x^4-\frac{\frac{1}{2}q_0l}{6}x^3+\frac{\frac{q_0l^2}{6}}{2}x^2$$

$$EIw=\frac{-q_0l}{12}x^3+\frac{q_0l^2}{12}x^2+q_0(\frac{x^4}{24}-\frac{x^5}{120l})$$

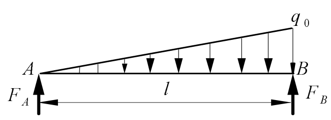
$$EI\theta=-\frac{q_0l}{4}x^2+\frac{q_0l^2}{6}x^2+q_0(\frac{x^3}{6}-\frac{x^4}{24l})$$

$$w_{\max}=w(l)=\frac{1}{EI}[-\frac{q_0l}{12}\cdot l^3+\frac{q_0l^2}{12}l^2+q_0(\frac{l^4}{24}-\frac{l^5}{120l})]=\frac{q_0l^4}{30EI}$$

$$\theta_B=\theta(l)=\frac{1}{EI}[-\frac{q_0l}{4}\cdot l^2+\frac{q_0l^2}{6}l+q_0(\frac{l^3}{6}-\frac{l^4}{24l})]=\frac{q_0l^3}{24EI}$$

5-23　试用初参数法验算附录 IV 中第 9 项中梁跨中截面的挠度及支座处截面的转角表达式。

解： $\sum M_B=0,F_Al-\frac{1}{2}q_0l\cdot\frac{l}{3}=0$，$F_A=\frac{1}{6}q_0l$



$$EIw=\iiiint\frac{q_0}{l}x\,\mathrm{d}\,x^4-\frac{\frac{1}{6}q_0l}{6}x^3+EI\theta_Ax$$

$$EIw=\frac{q_0l^5}{120l}-\frac{q_0l}{36}x^3+EI\theta_Ax$$

由 $w(l)=0$，$\frac{q_0l^5}{120l}-\frac{q_0l}{36}\cdot l^3+EI\theta_A\cdot l=0$

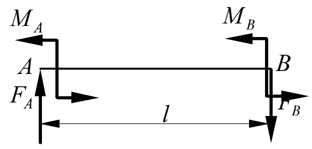
$$\theta_A=\frac{7q_0l^3}{360EI}$$

$$EI\theta=EIw'=\frac{q_0x^4}{24l}-\frac{q_0l}{12}x^2+EI\theta_A$$

$$\theta_B = \frac{1}{EI}(\frac{q_0 l^4}{24l} - \frac{q_0 l}{12}\cdot l^2 + \frac{7q_0 l^3}{360}) = \frac{-q_0 l^3}{45EI}$$

$$w_C = \frac{1}{EI}[\frac{q_0}{120l}(\frac{l}{2})^5 - \frac{q_0 l}{36}(\frac{l}{2})^3 + EI\cdot\frac{7q_0 l^3}{360EI}\cdot\frac{l}{2}] = \frac{5q_0 l^4}{768EI}$$

5-24　在简支梁的两支座截面上分别承受外力偶矩M_A和M_B，如题 5-9 图所示。已知该梁的弯曲刚度为 EI，试用初参数法求θ_A。



解： $\sum M_B = 0$，$F_A = \dfrac{M_A + M_B}{l}$

$$EIw = -\frac{1}{6}\cdot\frac{M_A + M_B}{l}x^3 + \frac{M_A}{2}x^2 + EI\theta_A$$

$$w(l) = 0, -\frac{M_A + M_B}{6l}l^3 + \frac{M_A}{2}l^2 + EI\theta_A = 0$$

$$\theta_A = \frac{l}{6EI}(M_B - 2M_A)$$

5-25　松木桁条的横截面为圆形，跨长为 4m，两端可视为简支，全跨上作用有集度为$q = 1.82\,\text{kN/m}$的均布荷载。已知松木的许用应力$[\sigma] = 10\,\text{MPa}$，弹性模量$E = 10\,\text{GPa}$。桁条的许可相对挠度为$\left[\dfrac{w}{l}\right] = \dfrac{1}{200}$。试求桁条横截面所需的直径。（桁条可视为等直圆木梁计算，直径以跨中为准。）

解： 均布荷载简支梁，其危险截面位于跨中点，最大弯矩为$M_{\max} = \dfrac{1}{8}ql^2$，根据强度条件有

$$\sigma_{\max} = \frac{M_{\max}}{W_z} = \frac{32M_{\max}}{\pi d^3} \le [\sigma]$$

从满足强度条件，得梁的直径为

$$d \ge \sqrt[3]{\frac{32M_{\max}}{\pi[\sigma]}} = \sqrt[3]{\frac{4ql^2}{\pi[\sigma]}} = \sqrt[3]{\frac{4\times1.82\times10^3\times4^2}{\pi\times10\times10^6}} = 0.155\,\text{m}$$

对圆木直径的均布荷载，简支梁的最大挠度$w_{\max}$为

$$w_{\max} = \frac{5ql^4}{384EI} = \frac{5ql^4}{384E\dfrac{\pi d^4}{64}} = \frac{5ql^4}{6E\pi d^4}$$

而相对挠度为　$$\frac{w_{\max}}{l} = \frac{5ql^3}{6E\pi d^4}$$

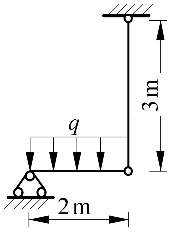
由梁的刚度条件有　$$\frac{w_{\max}}{l} = \frac{5ql^3}{6E\pi d^4} \le \left[\frac{w}{l}\right]$$

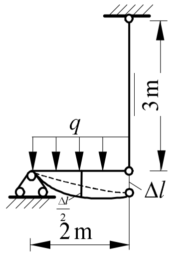
为满足梁的刚度条件，梁的直径有

$$d \ge \sqrt[4]{\frac{5ql^3}{6E\pi\left[\dfrac{w}{l}\right]}} = \sqrt[4]{\frac{5\times1.82\times10^3\times4^3}{6\times10\times10^9\pi\times\dfrac{1}{200}}} = 0.158\,\text{m}$$

由上可见，为保证满足梁的强度条件和刚度条件，圆木直径需大于$158\,\text{mm}$。

5-26　图示木梁的右端由钢拉杆支承。已知梁的横截面为边长等于 0.20 m 的正方形，$q=40\ \mathrm{kN/m}$，$E_1=10\ \mathrm{GPa}$；钢拉杆的横截面面积 $A_2=250\ \mathrm{mm}^2, E_2=210\ \mathrm{GPa}$。试求拉杆的伸长 Δl 及梁中点沿铅垂方向的位移 $\varDelta$。





解：从木梁的静力平衡，易知钢拉杆受轴向拉力

$$F_{\mathrm{N}}=\frac{1}{2}ql_1=\frac{1}{2}\times 40\times 2=40\ \mathrm{kN}$$

于是拉杆的伸长 Δl 为

$$\Delta l=\frac{F_{\mathrm{N}}l_2}{E_2A_2}=\frac{40\times 10^3\times 3}{210\times 10^9\times 2.5\times 10^{-4}}$$

$$=2.28\times 10^{-3}\ \mathrm{m}=2.28\ \mathrm{mm}$$

木梁由于均布荷载产生的跨中挠度 w 为

$$w=\frac{5ql_1^4}{384E_1I}=\frac{5ql_1^4}{384E_1\dfrac{h^4}{12}}=\frac{5\times 40\times 10^3\times 2^4}{32\times 10\times 10^9\times 0.2^4}=6.25\times 10^{-3}\ \mathrm{m}=6.25\ \mathrm{mm}$$

梁中点的铅垂位移 $\varDelta$ 等于因拉杆伸长引起梁中点的刚性位移 $\dfrac{\Delta l}{2}$ 与中点挠度 w 的和，即

$$\varDelta=\frac{\Delta l}{2}+w=\frac{2.28}{2}+6.25=7.39\ \mathrm{mm}$$