# 算例 3-004

## 平面 – 厚墙圆筒

### 问题描述

本例为一个厚墙形成的圆筒，按 MacNeal and Harder 1985 所述，承受 1ksi 的内压。本例中分别考虑平面应变和平面应力情况。用 5x1x1 网格模拟圆筒中 10 度对应的一部分，厚度 1 英寸，内径 3 英寸，外径 9 英寸。将筒内侧的径向位移和径向、切向、长向应力分量与手算结果进行了比较。MacNeal and Harder 1985 中公布了平面应变状态下的径向位移值。

本例中使用四个不同泊松比和两个不同网格建模。下表列出了所使用的模型。

| 模型 | 泊松比 | 网格 |
|---|---|---|
| A | 0.3 | 5 x 1 x 1 |
| B | 0.49 | 5 x 1 x 1 |
| C | 0.499 | 5 x 1 x 1 |
| D | 0.4999 | 5 x 1 x 1 |
| E | 0.49 | 20 x 1 x 1 |
| F | 0.49 | 160 x 1 x 1 |

六个模型每个运行两次：一次使用平面应变单元，一次使用平面应力单元。

分析中 $U_x$ 和 $U_y$ 自由度为活动的。所有其他自由度都是不活动的。本例中对所有模型使用不相容弯曲模式选项。

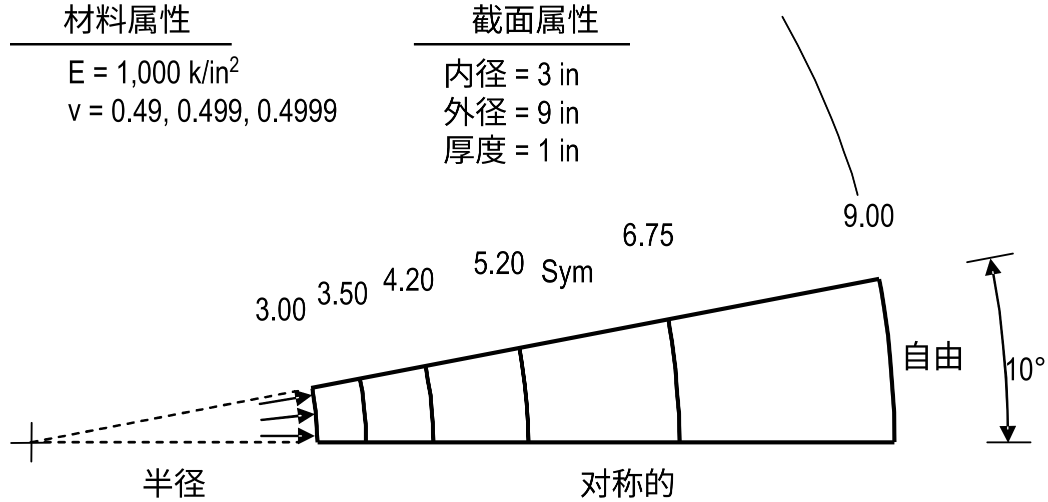
## 几何特性、属性和荷载

**材料属性**

E = 1,000 k/in$^2$
v = 0.49, 0.499, 0.4999

**截面属性**

内径 = 3 in
外径 = 9 in
厚度 = 1 in




## 校验的 SAP2000 的技术特色

- 使用平面应力单元的分析
- 使用平面应变单元的分析
- 平面表面压力荷载

## 结果比较

使用从 Timoshenko 1956 中标题为 *Thermal Stresses in a Long, Hollow Cylinder* 的 44 章推导的公式来计算独立结果。MacNeal and Harder 1985 中给出了泊松比为 0.49、0.499 和 0.4999 时的径向位移独立结果。

平面应力结果基于 Roark and Young 1975 第 504 页表 32 的 1a 项的公式。

平面应变

| 输出参数 | 模型、泊松比和网格 | SAP2000 | 独立结果 | 差异百分比* |
|---|---|---|---|---|
| 径向位移 in 内表面 | **A**, ν = 0.3, 5x1x1 | 0.004539 | 0.004582 | -1% |
| | **B**, ν = 0.49, 5x1x1 | 0.004971 | 0.005040 | -1% |
| | **C**, ν = 0.499, 5x1x1 | 0.004990 | 0.005060 | -1% |
| | **D**, ν = 0.4999, 5x1x1 | 0.004992 | 0.005062 | -1% |
| | **E**, ν = 0.49, 20x1x1 | 0.005036 | 0.005040 | 0% |
| | **F**, ν = 0.49, 160x1x1 | 0.005040 | 0.005040 | 0% |
| 径向应力 ksi 内表面 | **A**, ν = 0.3, 5x1x1 | -0.818 | -1.000 | -18% (-15%) |
| | **B**, ν = 0.49, 5x1x1 | -0.816 | -1.000 | -18% (-15%) |
| | **C**, ν = 0.499, 5x1x1 | -0.816 | -1.000 | -18% (-15%) |
| | **D**, ν = 0.4999, 5x1x1 | -0.816 | -1.000 | -18% (-15%) |
| | **E**, ν = 0.49, 20x1x1 | -0.937 | -1.000 | -6% (-5%) |
| | **F**, ν = 0.49, 160x1x1 | -0.960 | -1.000 | -4% (-3%) |
| 切向应力 ksi 内表面 | **A**, ν = 0.3, 5x1x1 | 1.277 | 1.250 | 2% |
| | **B**, ν = 0.49, 5x1x1 | 1.348 | 1.250 | 8% |
| | **C**, ν = 0.499, 5x1x1 | 1.352 | 1.250 | 8% |
| | **D**, ν = 0.4999, 5x1x1 | 1.353 | 1.250 | 8% |
| | **E**, ν = 0.49, 20x1x1 | 1.274 | 1.250 | 2% |
| | **F**, ν = 0.49, 160x1x1 | 1.255 | 1.250 | 0% |
| 纵向应力 ksi 内表面 | **A**, ν = 0.3, 5x1x1 | 0.138 | 0.075 | 84% (5%) |
| | **B**, ν = 0.49, 5x1x1 | 0.260 | 0.123 | 111% (11%) |
| | **C**, ν = 0.499, 5x1x1 | 0.268 | 0.125 | 114% (11%) |
| | **D**, ν = 0.4999, 5x1x1 | 0.268 | 0.125 | 114% (11%) |
| | **E**, ν = 0.49, 20x1x1 | 0.165 | 0.123 | 34% (3%) |
| | **F**, ν = 0.49, 160x1x1 | 0.144 | 0.123 | 17% (2%) |

* 括号中的差异百分比是相对于最大理论切向应力 1.250 ksi 的。

## 平面应力

| 输出参数 | 模型、泊松比和网格 | SAP2000 | 独立结果 | 差异百分比* |
|---|---|---|---|---|
| 径向位移 in 内表面 | **A**, ν = 0.3, 5x1x1 | 0.004610 | 0.004650 | -1% |
| | **B**, ν = 0.49, 5x1x1 | 0.005168 | 0.005220 | -1% |
| | **C**, ν = 0.499, 5x1x1 | 0.005194 | 0.005247 | -1% |
| | **D**, ν = 0.4999, 5x1x1 | 0.005197 | 0.005250 | -1% |
| | **E**, ν = 0.49, 20x1x1 | 0.005217 | 0.005220 | 0% |
| | **F**, ν = 0.49, 160x1x1 | 0.005220 | 0.005220 | 0% |
| 径向应力 ksi 内表面 | **A**, ν = 0.3, 5x1x1 | -0.819 | -1.000 | -18% (-14%) |
| | **B**, ν = 0.49, 5x1x1 | -0.818 | -1.000 | -18% (-15%) |
| | **C**, ν = 0.499, 5x1x1 | -0.818 | -1.000 | -18% (-15%) |
| | **D**, ν = 0.4999, 5x1x1 | -0.818 | -1.000 | -18% (-15%) |
| | **E**, ν = 0.49, 20x1x1 | -0.938 | -1.000 | -6% (-5%) |
| | **F**, ν = 0.49, 160x1x1 | -0.960 | -1.000 | -4% (-3%) |
| 切向应力 ksi 内表面 | **A**, ν = 0.3, 5x1x1 | 1.259 | 1.250 | 1% |
| | **B**, ν = 0.49, 5x1x1 | 1.285 | 1.250 | 3% |
| | **C**, ν = 0.499, 5x1x1 | 1.286 | 1.250 | 3% |
| | **D**, ν = 0.4999, 5x1x1 | 1.287 | 1.250 | 3% |
| | **E**, ν = 0.49, 20x1x1 | 1.253 | 1.250 | 0% |
| | **F**, ν = 0.49, 160x1x1 | 1.244 | 1.250 | 0% |

* 括号中的差异百分比是相对于最大理论切向应力 1.250 ksi 的。

**计算模型文件：** Example 3-004a-strain, Example 3-004a-stress
Example 3-004b- strain, Example 3-004b- stress
Example 3-004c- strain, Example 3-004c- stress
Example 3-004d- strain, Example 3-004d- stress
Example 3-004e- strain, Example 3-004e- stress
Example 3-004f- strain, Example 3-004f- stress

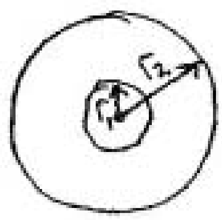
## 结论

与独立结果相比，当使用充分的剖分时，SAP2000 的结果可以接受。当网格细化时，SAP2000 解收敛于理论解。

## 手算过程

### Plane Strain Solution

Reference: Timoshenko, 1956
Strength of Materials, Part II
Chapter VI, Section 44




U = radial displ

$$\left.\begin{aligned} \varepsilon_r &= \frac{du}{dr} \\ \varepsilon_t &= \frac{u}{r} \\ \varepsilon_z &= 0 \text{ (plane strain)} \end{aligned}\right\} \text{Timoshenko Eqn (a)}$$

$$\left.\begin{aligned} &\Delta = \text{Increase in unit volume} \\ &\Delta = \varepsilon_r + \varepsilon_t + \varepsilon_z = \varepsilon_r + \varepsilon_t \end{aligned}\right\} \text{Timoshenko Eqn (c)}$$

$$\left.\begin{aligned} \sigma_r &= \frac{E}{1+v}\left(\varepsilon_r + \frac{v}{1-2v}\Delta\right) \\ \sigma_t &= \frac{E}{1+v}\left(\varepsilon_t + \frac{v}{1-2v}\Delta\right) \\ \sigma_z &= \frac{E}{1+v}\left(\varepsilon_z + \frac{v}{1-2v}\Delta\right) = \frac{Ev\Delta}{(1+v)(1-2v)} \end{aligned}\right\} \text{Timoshenko Eqn (d)}$$

The equation for radial displacement u is:

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(ru)\right) = 0$$

Timoshenko Eqn (202)

Integrate once to get

$$\frac{1}{r}\frac{d}{dr}(ru) = a$$

$$\frac{d}{dr}(ru) = ar$$

Integrate again to get

$$ru = ar^2 + b$$

$$\boxed{u = ar + \frac{b}{r}}$$

a & b are constants that will be derived later

$$\varepsilon_r = \frac{du}{dr} = a - \frac{b}{r^2}$$

$$\varepsilon_t = \frac{u}{r} = a + \frac{b}{r^2}$$

$$\sigma_r = \frac{E}{1+\nu}\left(\varepsilon_r + \frac{\nu}{1-2\nu}(\varepsilon_r + \varepsilon_t)\right)$$

$$= \frac{E}{1+\nu}\left(\varepsilon_r\left(1+\frac{\nu}{1-2\nu}\right) + \frac{\nu\varepsilon_t}{1-2\nu}\right)$$

$$= \frac{E}{1+\nu}\left(\varepsilon_r\left(\frac{1-\nu}{1-2\nu}\right) + \frac{\nu\varepsilon_t}{1-2\nu}\right)$$

$$= \frac{E}{(1+\nu)(1-2\nu)}\left(\varepsilon_r(1-\nu) + \nu\varepsilon_t\right)$$

$$= \frac{E}{(1+\nu)(1-2\nu)}\left(\left(a-\frac{b}{r^2}\right)(1-\nu) + \nu\left(a+\frac{b}{r^2}\right)\right)$$

$$= \frac{E}{(1+\nu)(1-2\nu)}\left(a - a\nu - \frac{b}{r^2} + \frac{\nu b}{r^2} + \nu a + \frac{\nu b}{r^2}\right)$$

$$= \frac{E}{(1+\nu)(1-2\nu)}\left(a - \frac{b}{r^2} + \frac{2b\nu}{r^2}\right)$$

$$\sigma_r = \frac{E}{(1+\nu)(1-2\nu)}\left(a - (1-2\nu)\frac{b}{r^2}\right)$$

At $r = r_2$, $\sigma_r = 0$, thus

$$\boxed{a = (1-2\nu)\frac{b}{r_2^2}}$$

$$\sigma_r = \frac{E}{(1+v)(1-2v)}\left((1-2v)\frac{b}{r_2^2} - (1-2v)\frac{b}{r^2}\right)$$

$$\boxed{\sigma_r = \frac{E}{1+v}\left(\frac{b}{r_2^2} - \frac{b}{r^2}\right)}$$

At $r = r_1$, $\sigma_r = -P$, thus

$$-P = \frac{E}{1+v}\left(\frac{1}{r_2^2} - \frac{1}{r^2}\right)b$$

$$\boxed{b = \frac{-P(1+v)}{E\left(\frac{1}{r_2^2} - \frac{1}{r^2}\right)}}$$

$$\sigma_t = \frac{E}{1+v}\left(\varepsilon_t + \frac{v}{1-v}(\varepsilon_r + \varepsilon_t)\right)$$

$$= \frac{E}{(1+v)(1-2v)}\left((1-v)\varepsilon_t + v\varepsilon_r\right)$$

$$= \frac{E}{(1+v)(1-2v)}\left((1-v)\left(a + \frac{b}{r^2}\right) + v\left(a - \frac{b}{r^2}\right)\right)$$

$$\sigma_t = \frac{E}{(1+v)(1-2v)}\left(a + \frac{b}{r^2} - av - \frac{vb}{r^2} + av - \frac{vb}{r^2}\right)$$

$$\sigma_t = \frac{E}{(1+\nu)(1-2\nu)}\left(a + (1-2\nu)\frac{b}{r^2}\right)$$

$$= \frac{E}{(1+\nu)(1-2\nu)}\left((1-2\nu)\frac{b}{r_2^2} + (1-2\nu)\frac{b}{r^2}\right)$$

$$\boxed{\sigma_t = \frac{E}{1+\nu}\left(\frac{b}{r_2^2} + \frac{b}{r^2}\right)}$$

$$\sigma_z = \frac{E\nu(\varepsilon_r + \varepsilon_t)}{(1+\nu)(1-2\nu)}$$

$$= \frac{E\nu}{(1+\nu)(1-2\nu)}\left(a - \frac{b}{r^2} + a + \frac{b}{r^2}\right)$$

$$= \frac{2E\nu a}{(1+\nu)(1-2\nu)}$$

$$= \frac{2E\nu}{(1+\nu)(1-2\nu)} \times (1-2\nu)\frac{b}{r_2^2}$$

$$\boxed{\sigma_z = \frac{2E\nu b}{(1+\nu)\, r_2^2}}$$

Summary of derived equations for plane strain solution.

Constants

$$b = \frac{-P(1+v)}{E\left(\frac{1}{r_2^2} - \frac{1}{r^2}\right)}$$

$$a = (1-2v)\frac{b}{r_2^2}$$

Radial Displacement

$$u = ar + \frac{b}{r}$$

Stress

$$\sigma_r = \frac{E}{1+v}\left(\frac{b}{r_2^2} - \frac{b}{r^2}\right)$$

$$\sigma_t = \frac{E}{1+v}\left(\frac{b}{r_2^2} + \frac{b}{r^2}\right)$$

$$\sigma_z = \frac{2Evb}{(1+v)\,r_2^2}$$

Calculate constants for different poisson's ratios

| $\nu$ | P | E | $r_2$ | r | b | a |
|---|---|---|---|---|---|---|
| 0.3 | 1 | 1000 | 9 | 3 | 0.0131625 | 0.000065 |
| 0.49 | 1 | 1000 | 9 | 3 | 0.01508625 | 0.000003725 |
| 0.499 | 1 | 1000 | 9 | 3 | 0.015177375 | 3.7475E-07 |
| 0.4999 | 1 | 1000 | 9 | 3 | 0.015186488 | 3.74975E-08 |

Calculate radial displacement

| $\nu$ | $\Delta_r$ (in) |
|---|---|
| 0.3 | 0.004583 |
| 0.49 | 0.005040 |
| 0.499 | 0.005060 |
| 0.4999 | 0.005062 |

Calculate stresses

| $\nu$ | $\sigma_r$ (ksi) | $\sigma_t$ (ksi) | $\sigma_z$ (ksi) |
|---|---|---|---|
| 0.3 | −1 | 1.25 | 0.075 |
| 0.49 | −1 | 1.25 | 0.1225 |
| 0.499 | −1 | 1.25 | 0.12475 |
| 0.4999 | −1 | 1.25 | 0.124975 |

Plane Stress Solution

Reference: Roark and Young, 1975
Formulas For Stress and Strain
Table 32, item 1a, pg 504

$\sigma_r$ @ inner surface $= -P$

$\sigma_t$ @ inner surface $= P \dfrac{r_2^2 + r_1^2}{r_2^2 - r_1^2}$

$\sigma_z = 0$

$\Delta$ @ inner surface $= \dfrac{P r_1}{E}\left(\dfrac{r_2^2 + r_1^2}{r_2^2 - r_1^2} + \nu\right)$

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$P = 1$ Ksi
$r_2 = 9$ in
$r_1 = 3$ in
$E = 1000$ Ksi
$\nu = 0.3, 0.49, 0.499, 0.4999$

$\sigma_r = -1$ Ksi

$\sigma_t = 1 \times \dfrac{9^2 + 3^2}{9^2 - 3^2} = 1.25$ Ksi

$$\Delta_r = \frac{1 \times 3}{1000}\left(\frac{9^2+3^2}{9^2-3^2} + \nu\right) = 0.003(1.25+\nu)$$

| $\nu$ | $\Delta_r$ |
|---|---|
| 0.3 | 0.004650 in |
| 0.49 | 0.005220 in |
| 0.499 | 0.005247 in |
| 0.4999 | 0.005250 in |