# 算例 5-008

## 实体单元 – 厚墙柱面

### 问题描述

在本例中，一个在平面应力状态下厚墙象 MacNeal and Harder 1985 所建议的施加 1-ksi 内部压强。一个 10 度的柱面片段、1-inch 厚，3-inch 内径和 9-inch 外径。并进行了 5X1X1 的单元剖分。柱面内表面的径向位移，径向、切向和纵向应力分量的程序计算结果和手算结果进行了对比.注意径向位移数值也在 MacNeal and Harder 1985 进行了发表.

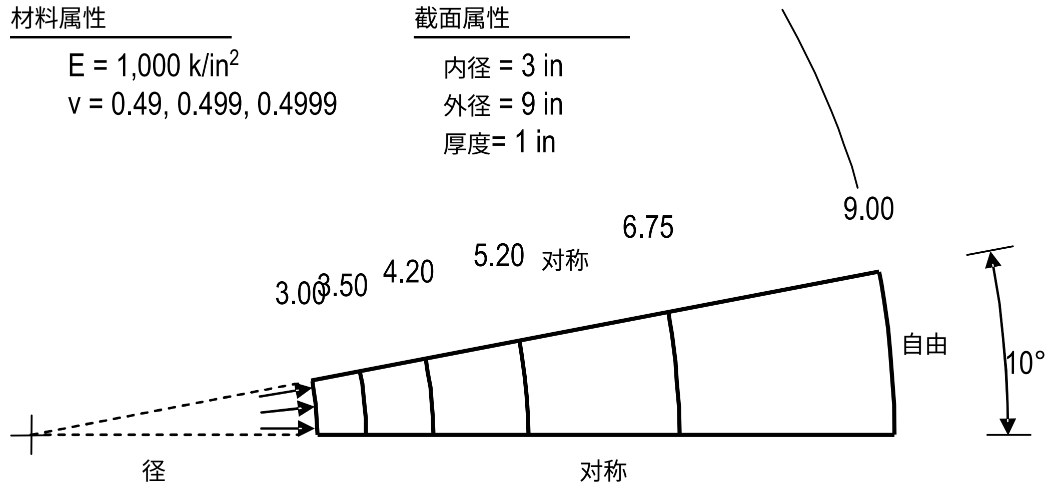
本例题中考虑了使用四个不同的泊松比和两个不同的剖分形式的六个模型,下表列出了所使用的模型:

| 模型 | 泊松比 | 剖分 |
|---|---|---|
| A | 0.3 | 5 x 1 x 1 |
| B | 0.49 | 5 x 1 x 1 |
| C | 0.499 | 5 x 1 x 1 |
| D | 0.4999 | 5 x 1 x 1 |
| E | 0.49 | 20 x 1 x 1 |
| F | 0.49 | 160 x 1 x 1 |

对于模型轴对称条件下,沿着模型对称边所应用的柱面片段的顶部和底部水平约束形式中弯矩被约束.为了实施这一约束,节点的局部坐标轴方向,节点 2 轴垂直于柱面的对称边.然后横向约束将在节点的 2 轴方向进行施加.

平面应力条件是通过设置分析 Uz 自由度不激活的状态来实现的。

## 几何、属性和荷载




## 所测试的 SAP2000 技术要点：

- 使用实体元分析
- 实体元表面压力荷载
- 节点局部坐标

## 结果比较

独立验算所使用的公式为文章 44, Timoshenko 1956 年发表的题为 *Thermal Stresses in a Long, Hollow Cylinder* 的文章。径向位移独立计算所使用的泊松比分别为 0.49, 0.499 和 0.4999，它们也是 MacNeal and Harder 在 1985 年发表的。径向位移手算基于的 Timoshenko 1956 公式与 MacNeal and Harder 1985 年给出的公式是一样的。

| 输出参数 | 模型、泊松比和剖分 | SAP2000 | 独立结果 | 误差* |
|---|---|---|---|---|
| 径向位移 in 在内表面 | **A**, ν = 0.3, 5x1x1 | 0.004539 | 0.004582 | -1% |
| | **B**, ν = 0.49, 5x1x1 | 0.004971 | 0.005040 | -1% |
| | **C**, ν = 0.499, 5x1x1 | 0.004990 | 0.005060 | -1% |
| | **D**, ν = 0.4999, 5x1x1 | 0.004992 | 0.005062 | -1% |
| | **E**, ν = 0.49, 20x1x1 | 0.005036 | 0.005040 | 0% |
| | **F**, ν = 0.49, 160x1x1 | 0.005041 | 0.005040 | 0% |
| 径向应力 ksi 在内表面 | **A**, ν = 0.3, 5x1x1 | -0.834 | -1.000 | -17% (-13%) |
| | **B**, ν = 0.49, 5x1x1 | -0.833 | -1.000 | -17% (-13%) |
| | **C**, ν = 0.499, 5x1x1 | -0.833 | -1.000 | -17% (-13%) |
| | **D**, ν = 0.4999, 5x1x1 | -0.833 | -1.000 | -17% (-13%) |
| | **E**, ν = 0.49, 20x1x1 | -0.954 | -1.000 | -5% (-4%) |
| | **F**, ν = 0.49, 160x1x1 | -0.994 | -1.000 | -1% (0%) |
| 切向应力 ksi 在内表面 | **A**, ν = 0.3, 5x1x1 | 1.293 | 1.250 | 3% |
| | **B**, ν = 0.49, 5x1x1 | 1.364 | 1.250 | 9% |
| | **C**, ν = 0.499, 5x1x1 | 1.369 | 1.250 | 10% |
| | **D**, ν = 0.4999, 5x1x1 | 1.369 | 1.250 | 10% |
| | **E**, ν = 0.49, 20x1x1 | 1.291 | 1.250 | 3% |
| | **F**, ν = 0.49, 160x1x1 | 1.256 | 1.250 | 0% |
| 纵向应力 ksi 在内表面 | **A**, ν = 0.3, 5x1x1 | 0.138 | 0.075 | 84% (5%) |
| | **B**, ν = 0.49, 5x1x1 | 0.260 | 0.123 | 111% (11%) |
| | **C**, ν = 0.499, 5x1x1 | 0.268 | 0.125 | 114% (11%) |
| | **D**, ν = 0.4999, 5x1x1 | 0.268 | 0.125 | 114% (11%) |
| | **E**, ν = 0.49, 20x1x1 | 0.165 | 0.123 | 34% (3%) |
| | **F**, ν = 0.49, 160x1x1 | 0.128 | 0.123 | 4% (0%) |

* 括号内的不同百分率是相对于理论最大切向应力 1.250 ksi 相比的。

**计算模型文件:** Example 5-008a, Example 5-008b, Example 5-008c
Example 5-008d, Example 5-008e, Example 5-008f

## 结论

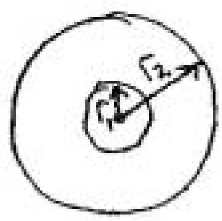
SAP2000 结果给出了在足够的剖分细度的情况下，与手算结果之间可以接受的差异。当剖分的细度进一步细化时，SAP2000 将给出更接近理论值的结果。

## HAND CALCULATION

Plane Strain Solution

Reference: Timoshenko, 1956
Strength of Materials, Part II
Chapter VI, Section 44

$U$ = radial displ




$$\left.\begin{aligned} \varepsilon_r &= \frac{du}{dr} \\ \varepsilon_t &= \frac{u}{r} \\ \varepsilon_z &= 0 \text{ (plane strain)} \end{aligned}\right\} \text{Timoshenko Eqn (a)}$$

$$\left.\begin{aligned} &\Delta = \text{Increase in unit volume} \\ &\Delta = \varepsilon_r + \varepsilon_t + \varepsilon_z = \varepsilon_r + \varepsilon_t \end{aligned}\right\} \text{Timoshenko Eqn (c)}$$

$$\left.\begin{aligned} \sigma_r &= \frac{E}{1+\nu}\left(\varepsilon_r + \frac{\nu}{1-2\nu}\Delta\right) \\ \sigma_t &= \frac{E}{1+\nu}\left(\varepsilon_t + \frac{\nu}{1-2\nu}\Delta\right) \\ \sigma_z &= \frac{E}{1+\nu}\left(\varepsilon_z + \frac{\nu}{1-2\nu}\Delta\right) = \frac{E\nu\Delta}{(1+\nu)(1-2\nu)} \end{aligned}\right\} \text{Timoshenko Eqn (d)}$$

The equation for radial displacement u is:

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(ru)\right)=0$$

Timoshenko Eqn (202)

Integrate once to get

$$\frac{1}{r}\frac{d}{dr}(ru) = a$$

$$\frac{d}{dr}(ru) = ar$$

Integrate again to get

$$ru = ar^2 + b$$

$$\boxed{u = ar + \frac{b}{r}}$$

a & b are constants that will be derived later

$$\epsilon_r = \frac{du}{dr} = a - \frac{b}{r^2}$$

$$\epsilon_t = \frac{u}{r} = a + \frac{b}{r^2}$$

$$\sigma_r = \frac{E}{1+v}\left(\varepsilon_r + \frac{v}{1-2v}\left(\varepsilon_r + \varepsilon_t\right)\right)$$

$$= \frac{E}{1+v}\left(\varepsilon_r\left(1 + \frac{v}{1-2v}\right) + \frac{v\varepsilon_t}{1-2v}\right)$$

$$= \frac{E}{1+v}\left(\varepsilon_r\left(\frac{1-v}{1-2v}\right) + \frac{v\varepsilon_t}{1-2v}\right)$$

$$= \frac{E}{(1+v)(1-2v)}\left(\varepsilon_r(1-v) + v\varepsilon_t\right)$$

$$= \frac{E}{(1+v)(1-2v)}\left(\left(a - \frac{b}{r^2}\right)(1-v) + v\left(a + \frac{b}{r^2}\right)\right)$$

$$= \frac{E}{(1+v)(1-2v)}\left(a - av - \frac{b}{r^2} + \frac{vb}{r^2} + va + \frac{vb}{r^2}\right)$$

$$= \frac{E}{(1+v)(1-2v)}\left(a - \frac{b}{r^2} + \frac{2bv}{r^2}\right)$$

$$\sigma_r = \frac{E}{(1+v)(1-2v)}\left(a - (1-2v)\frac{b}{r^2}\right)$$

At $r = r_2$, $\sigma_r = 0$, thus

$$\boxed{a = (1-2v)\frac{b}{r_2^2}}$$

$$\sigma_r = \frac{E}{(1+v)(1-2v)}\left((1-2v)\frac{b}{r_2^2} - (1-2v)\frac{b}{r^2}\right)$$

$$\boxed{\sigma_r = \frac{E}{1+v}\left(\frac{b}{r_2^2} - \frac{b}{r^2}\right)}$$

At $r = r_1$, $\sigma_r = -P$, thus

$$-P = \frac{E}{1+v}\left(\frac{1}{r_2^2} - \frac{1}{r^2}\right)b$$

$$\boxed{b = \frac{-P(1+v)}{E\left(\frac{1}{r_2^2} - \frac{1}{r^2}\right)}}$$

$$\sigma_t = \frac{E}{1+v}\left(\varepsilon_t + \frac{v}{1-v}(\varepsilon_r + \varepsilon_t)\right)$$

$$= \frac{E}{(1+v)(1-2v)}\left((1-v)\varepsilon_t + v\varepsilon_r\right)$$

$$= \frac{E}{(1+v)(1-2v)}\left((1-v)\left(a + \frac{b}{r^2}\right) + v\left(a - \frac{b}{r^2}\right)\right)$$

$$\sigma_t = \frac{E}{(1+v)(1-2v)}\left(a + \frac{b}{r^2} - av - \frac{vb}{r^2} + av - \frac{vb}{r^2}\right)$$

$$\sigma_t = \frac{E}{(1+v)(1-2v)}\left(a + (1-2v)\frac{b}{r^2}\right)$$

$$= \frac{E}{(1+v)(1-2v)}\left((1-2v)\frac{b}{r_2^2} + (1-2v)\frac{b}{r^2}\right)$$

$$\boxed{\sigma_t = \frac{E}{1+v}\left(\frac{b}{r_2^2} + \frac{b}{r^2}\right)}$$

$$\sigma_z = \frac{E v(\varepsilon_r + \varepsilon_t)}{(1+v)(1-2v)}$$

$$= \frac{Ev}{(1+v)(1-2v)}\left(a - \frac{b}{r^2} + a + \frac{b}{r^2}\right)$$

$$= \frac{2Eva}{(1+v)(1-2v)}$$

$$= \frac{2Ev}{(1+v)(1-2v)} \times (1-2v)\frac{b}{r_2^2}$$

$$\boxed{\sigma_z = \frac{2Evb}{(1+v)\,r_2^2}}$$

Summary of derived equations for plane strain solution.

Constants

$$b = \frac{-P(1+v)}{E\left(\frac{1}{r_2^2} - \frac{1}{r^2}\right)}$$

$$a = (1-2v)\frac{b}{r_2^2}$$

Radial Displacement

$$u = ar + \frac{b}{r}$$

Stress

$$\sigma_r = \frac{E}{1+v}\left(\frac{b}{r_2^2} - \frac{b}{r^2}\right)$$

$$\sigma_t = \frac{E}{1+v}\left(\frac{b}{r_2^2} + \frac{b}{r^2}\right)$$

$$\sigma_z = \frac{2Evb}{(1+v)\,r_2^2}$$

Calculate constants for different poisson's ratios

| ν | P | E | $r_2$ | r | b | a |
|---|---|---|---|---|---|---|
| 0.3 | 1 | 1000 | 9 | 3 | 0.0131625 | 0.000065 |
| 0.49 | 1 | 1000 | 9 | 3 | 0.01508625 | 0.000003725 |
| 0.499 | 1 | 1000 | 9 | 3 | 0.015177375 | 3.7475E-07 |
| 0.4999 | 1 | 1000 | 9 | 3 | 0.015186488 | 3.74975E-08 |

Calculate radial displacement

| ν | $\Delta_r$ (in) |
|---|---|
| 0.3 | 0.004583 |
| 0.49 | 0.005040 |
| 0.499 | 0.005060 |
| 0.4999 | 0.005062 |

Calculate stresses

| ν | $\sigma_r$ (ksi) | $\sigma_t$ (ksi) | $\sigma_z$ (ksi) |
|---|---|---|---|
| 0.3 | −1 | 1.25 | 0.075 |
| 0.49 | −1 | 1.25 | 0.1225 |
| 0.499 | −1 | 1.25 | 0.12475 |
| 0.4999 | −1 | 1.25 | 0.124975 |