ERRATA AND ADDENDA TO CHAPTERS 1-7 OF RUDIN'S
PRINCIPLES OF MATHEMATICAL ANALYSIS, 3rd Edition, 4th Printing (noted as of August, 2003)
For additional errata to **earlier printings**, see last page of these sheets.

☛ Note: If you don't want to write corrections into your text, you might put them on PostIts (or slivers of paper cut from PostIts) and insert these at the page in question. ☚

P.4, line 4: Change this line to ''(ii) If $\gamma \in S$ is an upper bound of E, then $\gamma \geq \alpha$'' for greater clarity

P.4 3rd line of Definition 1.10: A clearer statement would be, ''Every subset $E \subset S$ which is nonempty and bounded above has a supremum $\sup E$ in S.''

P.5, last 5 lines of proof of Theorem 1.11: Change these lines to:

If α were not a lower bound of B, there would be some $x \in B$ satisfying $x < \alpha$. This x would be an upper bound of L (by the preceding paragraph), contradicting our assumption that α is the *least* upper bound of L. So α is a lower bound of B. Now if y is any lower bound of B, then $y \in L$, so $y \leq \sup L = \alpha$; this shows that α is the *greatest* lower bound of B.

P.6, Proposition 1.14: Add

(e) $-(x+y) = (-x) + (-y)$.

Can you see how to prove this? I will either discuss it in class, or make it an exercise.

P.12, definitions of operations on the extended real numbers: Rudin should have noted the convention that $x+(+\infty)$ and $x+(-\infty)$ may be abbreviated $x+\infty$ and $x-\infty$ respectively, and mentioned that addition and multiplication are understood to be commutative on the extended reals, so that the definitions he gives also imply further cases like $+\infty+x = +\infty$. Finally, the three equations in (a), instead of having the common condition ''If x is real'', should be preceded by the respective conditions, ''If x is real or $+\infty$'', ''If x is real or $-\infty$'', and only in the last case simply ''If x is real''.

P.16, Theorem 1.37: Add one more part:

(g) *Assuming* $k > 0$, *there exists a vector* $\mathbf{u}$ *with* $|\mathbf{u}| = 1$ *such that* $\mathbf{u} \cdot \mathbf{x} = |\mathbf{x}|$.

Proof: If $\mathbf{x} \neq \mathbf{0}$ let $\mathbf{u} = |\mathbf{x}|^{-1}\mathbf{x}$; if $\mathbf{x} = \mathbf{0}$ let $\mathbf{u}$ be any vector with $|\mathbf{u}| = 1$.

P.19, middle: The author refers to the archimedean property of Q. This is *not* a consequence of Theorem 1.20(a); that would be circular reasoning. Rather, it is an elementary property of Q: Given x, $y \in Q$ with $x > 0$, we need to find an $n > y/x$. If $y/x < 0$, take $n = 1$; otherwise, write y/x as a fraction with positive denominator, and take for n any integer greater than its numerator.

P.36: After finishing the section of metric spaces, you might find the following discussion enlightening; but it is not required reading.

What is topology? Chapter 2 of Rudin is entitled ''Basic Topology'', but the chapter is about metric spaces, and the word ''topology'' does not appear in that chapter, nor in the index. What does it refer to?

Topology is a field of mathematics that includes the study of metric spaces as a special case. The key to the connection between *metric spaces* and the more general concept of a *topological space* is Theorem 2.24, parts (a) and (b) (p.34), which show that if we write T for the set of all open sets in a metric space, then the union of *any* family of members of T, and the intersection of any *finite* family of members of T, are also open sets. Families of sets with these properties come up in other contexts as well; so one makes

Definition. *A topological space* X means a pair (X, T), where X is a set, and T is a set of subsets of X which satisfies

(i) For any collection $\{G_\alpha\}$ with all $G_\alpha \in T$ one has $\bigcup_\alpha G_\alpha \in T$.

(ii) For any *finite* collection $\{G_\alpha\}$ with all $G_\alpha \in T$ one has $\bigcap_\alpha G_\alpha \in T$.

(iii) $\varnothing \in T$ and $X \in T$.

When T has been specified, and there is no danger of ambiguity, one simply speaks of ''the topological space X''. The sets in T are called the *open sets* of X.

(*Remark*: Condition (iii) can be omitted from this definition if one interprets conditions (i) and (ii) appropriately, since $\varnothing$ can be regarded as the union of the *empty* family of members of T, and if one interprets $\cap$ to refer to intersection *as subsets of* X, then X can likewise be regarded as the intersection of the empty family of members of T.)

Most of the concepts developed in Chapter 2 can be expressed in terms of open sets, hence they also make sense in a general topological space. For instance, a *closed set* G can be defined as a set whose complement G^c is open. (Under this definition, parts (c) and (d) of Theorem 2.24 clearly hold in any topological space.) A *limit point* of E can be defined as a point $p \in X$ such that every open subset of X which contains p contains a point of E other than p (cf. exercise 2.2:4 in the exercise packet). In terms of limit points, one can define *isolated point* and *perfect set*. (One can also check that Rudin's definition of ''closed set'' in terms of ''limit point'' yields, in this context, the class of sets we just defined to be closed.) One can define the *interior* E^o of a subset E to be the union of all open sets contained in E, and *interior point* to be a point of E^o. Rudin's definition of *compact set* (given at the beginning of the next section) will be stated in terms of open sets, so it, too, makes sense in this context.

Of the main concepts defined for general metric spaces in Chapter 2, there are two that don't have analogs in the general theory of topological spaces: those of ''neighborhood'' and of ''bounded subset''; these are among the features that distinguish the theory of metric spaces from the general theory of topological spaces. (Actually, topologists define a ''neighborhood'' of a point $p \in X$ to be any subset $E \subset X$ having p in its interior. In current usage, what Rudin calls a ''neighborhood'' is called an ''open ball'', so in modern language, it is the concept of ''open ball'' that is meaningful for metric spaces but not for general topological spaces.)

Why is it useful to study general topological spaces, and not just metric spaces? There are two reasons. One is that there are examples of topological spaces that don't arise from a metric. For instance, if X is any infinite set, one can take T to consist of all subsets $G \subset X$ such that either $G = \varnothing$ or $X - G$ is finite; this is a topology on X having properties that a topology arising from a metric can never have.

The other reason is that different metrics can correspond to the same topology, and it is sometimes important to realize that certain spaces are ''topologically the same'' even though they look different as metric spaces. As a trivial example, if d is any metric on a set X, then the metric d' given by $d'(x,y) = d(x,y)/2$ determines the same topology as d. For a less trivial example, let d be the ordinary metric on the segment $(-1,1) \subset R$, and let d'' be the metric defined by $d''(x,y) = |(\tan \pi x/2) - (\tan \pi y/2)|$. Since the function $\tan \pi x/2$ ''stretches'' the segment $(-1,1)$ to fill up the whole real line, d'' can be thought of as the metric on $(-1,1)$ induced by the ordinary metric on the ''stretched'' segment, the whole line. It is easy to show that the open subsets are the same under both metrics, namely the sets that can be written as unions of open intervals (in Rudin's language, as unions of segments); so we are talking about the same topology on our set $(-1,1)$; but under one metric, the set is bounded, and under the other, unbounded. (Similar ''stretching'' can change other commonplace shapes into very ''different-looking'' ones; in particular, there is a ''stretching'' that leads to the familiar quip that a topologist is a person who doesn't know the difference between a donut and a coffee-cup.)

A well-written standard introduction to topology is *General Topology* by John Kelley, Van Nostrand, 1955. (Kelley was a faculty member here at Berkeley.)

P.36, end of Definition 2.31: Add, ''If $\{G_\alpha\}$ is an open cover of E, then by a *subcover* we mean a subset of $\{G_\alpha\}$ which is also a cover of E.''

P.41, next-to-last paragraph of proof of Theorem 2.43: Replace this with the following three paragraphs, leaving the rest of the proof unchanged:

> Starting with V_1, we shall construct recursively a sequence of neighborhoods V_n with the following properties: (i_n) $V_n \cap P$ is not empty, (ii_n) If $n > 1$, then $\overline{V_n} \subset V_{n-1}$, ($\mathrm{iii}_n$) If $n > 1$, then $\mathbf{x}_{n-1} \notin \overline{V_n}$.
>
> Suppose inductively that V_n has been constructed. We claim that it contains a point $\mathbf{y} \in P$ other than $\mathbf{x}_n$. Indeed, by (i_n) it contains some point $\mathbf{z} \in P$, and if $\mathbf{z} \neq \mathbf{x}_n$ we are done. If $\mathbf{z} = \mathbf{x}_n$, note that since V_n is open, it contains some neighborhood U of $\mathbf{z}$, and because P is perfect, U contains some point $\mathbf{y} \in P$ other than $\mathbf{z}$. So let $\mathbf{y}$ be so chosen.
>
> Now, because V_n is open, it also contains a neighborhood of $\mathbf{y}$, say of radius r. Let us take for

V_{n+1} any neighborhood of $\mathbf{y}$ whose radius is both $< r$ and $< d(\mathbf{x}_n, \mathbf{y})$. From the first of these conditions one can deduce that (ii_{n+1}) holds and from the second that (iii_{n+1}) holds. Finally, the fact that $\mathbf{y} \in V_{n+1}$ gives (i_{n+1}), as required.

P.48 Theorem 3.2: Add

(e) *If* $\lim_{n\to\infty} p_n = p$, *and* $p_n \in E$ *for all* n, *then* $p \in \bar{E}$.

P.49, Theorem 3.3: *any number* should be *any complex number*.

P.51, first display in **Proof**: α_J should be α_j.

P.54, two lines before Definition 3.12: ‘‘(Theorem 2.41)’’ should be ‘‘(Theorem 2.41 and Theorem 3.10(a))’’.

P.54, two lines after Definition 3.12: Change ‘‘Theorem 3.11 implies also’’ to ‘‘Definition 3.12 implies’’.

P.59, line after second display: Change ‘‘For $\{s_n\}$’’ to ‘‘For the limit of the sequence $\{s_n\}$, if this exists,’’.

P.67, third line: $n \geq N$ should be $n > N$.

P.70, Theorem 3.42: Change the first word, ‘‘*Suppose*’’, to ‘‘*Let* $\{a_n\}$, $\{b_n\}$, $\{A_n\}$ *be as in Theorem 3.41, and suppose*’’.

P.73, 3rd line of Example 3.49: Change the initial word ‘‘and’’ to ‘‘where c_n is defined as in Definition 3.48, and if’’.

P.94, second display: Change ‘‘$= \lim_{t\to x} f(x)$’’ to ‘‘and when this holds, $\lim_{t\to x} f(x)$ is their common value’’.

P.98, line 7: Change ‘‘is not empty’’ to ‘‘has points other than x’’.

P.98, first line of Theorem 4.34: Change ‘‘*be defined*’’ to ‘‘*be real functions defined*’’.

P.105, proof of Theorem 5.5: In the line after (5), before ‘‘Let’’ add: ‘‘We also define $u(x) = 0$ and $v(y) = 0$; thus u is continuous at x, and v at y.’’ Then change the last two lines of the proof to ‘‘By Theorem 4.7 the right-hand side of (6) is continuous at $t = x$, where it has value $g'(y)f'(x)$, hence the left-hand side approaches this value as $t \to x$, which gives (3).’’

(Rudin’s proof skirts the point that makes proving the Chain Rule difficult: that $f(t)$ may take on the value $f(x)$ infinitely often in the neighborhood of x.)

P.108, title ‘‘**THE CONTINUITY OF DERIVATIVES**’’: Change to ‘‘**A RESTRICTION ON DISCONTINUITIES OF DERIVATIVES**’’.

P.109, display (13): Between ‘‘A’’ and ‘‘*as*’’ insert ‘‘(*a real number or* $\pm\infty$)’’.

P.109, add as a footnote to (18): ‘‘Note that by assumption, g' is nowhere 0 on (a, b). Hence by the Mean Value Theorem, $g(x) - g(y)$ is nonzero for distinct $x, y \in (a, b)$.’’

P.113, Theorem 5.19: For a simpler proof, use Theorem 1.37(g) (given in the note to p.16 above) to choose $\mathbf{u}$ so that $\mathbf{u}\cdot(\mathbf{f}(b) - \mathbf{f}(a)) = |\mathbf{f}(b) - \mathbf{f}(a)|$, and apply the Mean Value Theorem to $\mathbf{u}\cdot\mathbf{f}(t)$.

P.115, Exercise 13, display defining $f(x)$: x^a should be $|x|^a$.

P.118, first display: Change ‘‘$\genfrac{}{}{0pt}{}{1}{2}$’’ to ‘‘$\frac{1}{2}$’’.

P.123, Definition 6.3: All these partitions should be understood to be of a fixed interval $[a, b]$.

P.126 (17): $i-$ should be $i=$.

P.135, Theorem 6.25: Like Theorem 5.19 (see note to p.113 above), this can be proved more simply using Theorem 1.37(g). Can you see how?

P.141, 4th line from bottom: Change $[0, 2\pi]$ to $(0, 2\pi]$.

P.161, last line: change ‘‘Theorem 2.27’’ to the more precise ‘‘Theorem 2.27(a)’’.

P.166, line following first display: After ‘‘converges’’ add ‘‘on [0,1]’’.

P.336, line 3: Change ‘‘Pure Mathematics’’ to ‘‘A Course of Pure Mathematics’’.

ADDITIONAL CORRECTIONS TO MAKE IF YOU HAVE AN EARLIER PRINTING OF RUDIN’S *PRINCIPLES OF MATHEMATICAL ANALYSIS*

P.3, Definition 1.5, condition (ii): Change ‘‘and $y < x$’’ to ‘‘and $y < z$’’.

P.10, 2nd line of Theorem 1.21: Change ‘‘*one real* y’’ to ‘‘*one positive real* y’’.

P.10, 5th line of Proof: Change ‘‘$t^n < t$’’ to ‘‘$t^n \leq t$’’; and two lines later change ‘‘$t^n > t$’’ to ‘‘$t^n \geq t$’’.

P.17, Step 1, item (I): Change the comma to the word ‘‘and’’.

P.32, 4th line: After ‘‘$d(p,q) < r$’’, add ‘‘for some $r > 0$’’.

P.33, under Examples 2.21, description of (c): Change ‘‘finite set’’ to ‘‘finite nonempty set’’.

P.48, 3rd line of Theorem 3.2: Change ‘‘*all but finitely many of the terms of* $\{p_n\}$’’ to ‘‘p_n *for all but finitely many* n’’.

P.52, 3rd line from bottom: Change ‘‘subset’’ to ‘‘nonempty subset’’.

P.54, 5 lines above Definition 3.12: Change ‘‘$\{x_n\}$’’ (with italic x) to ‘‘$\{\mathbf{x}_n\}$’’ (with boldface $\mathbf{x}$), as on next line.

P.57, Example 3.18(b): Change ‘‘$(-1^n$’’ to ‘‘$(-1)^n$’’.

P.66, Theorem 3.34(b): Change ‘‘*for* n’’ to ‘‘*for all* n’’.

P.67, center of page, last of the four displayed lines beginning ‘‘lim sup’’: Change $\left(\frac{3}{2}\right)^n$ to $\frac{1}{2}\left(\frac{3}{2}\right)^n$.

P.72: Change equation under second Σ from ‘‘$n=k$’’ to ‘‘$k=n$’’ (as under first Σ).

P.75, 3rd line from bottom: Change ‘‘J to J’’ to ‘‘J onto J’’.

P.82, beginning of 4th line: Change ‘‘and bounded’’ to ‘‘nonempty and bounded’’.

P.84, five lines above Theorem 4.2: Change ‘‘appropriate norms’’ to ‘‘norms of differences’’.

P.98, first line of Definition 4.33 and first line of Theorem 4.34: Change ‘‘E’’ to ‘‘$E \subset R$’’.

P.123, line following first display: Change ‘‘are the same’’ to ‘‘mean the same thing’’.

P.162, line 2: Change ‘‘distincts point’’ to ‘‘distinct points’’.