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（影印版·原书第2版）

Calculus With Analytic Geometry

[美] 乔治 F. 西蒙斯（George F. Simmons） 著

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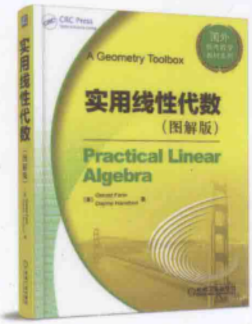

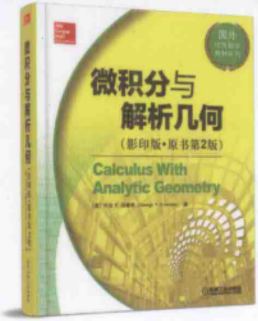

本书长期作为麻省理工学院的教材，为科学、工程或数学专业的学生特别设计了三学期的标准课程。附录中提供的大量内容，适合各个层面的教学。本书更加侧重微积分的具体应用，而不是复杂的证明。以问题为导向，提高学生处理实际问题的能力。

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ABOUT THE AUTHOR

George F. Simmons has the usual academic degrees (CalTech, Chicago, Yale) and taught at several colleges and universities before joining the faculty of Colorado College in 1962. He is also the author of *Introduction to Topology and Modern Analysis* (McGraw-Hill, 1963), *Differential Equations with Applications and Historical Notes* (McGraw-Hill, 1972, 2nd edition 1991), *Precalculus Mathematics in a Nutshell* (Janson Publications, 1981), and *Calculus Gems: Brief Lives and Memorable Mathematics* (McGraw-Hill, 1992).

When not working or talking or eating or drinking or cooking, Professor Simmons is likely to be traveling (Western and Southern Europe, Turkey, Israel, Egypt, Russia, China, Southeast Asia), trout fishing (Rocky Mountain states), playing pocket billiards, or reading (literature, history, biography and autobiography, science, and enough thrillers to achieve enjoyment without guilt). One of his personal heroes is the older friend who once said to him, "I should probably spend about an hour a week revising my opinions."

PREFACE TO THE INSTRUCTOR

It is a curious fact that people who write thousand-page textbooks still seem to find it necessary to write prefaces to explain their purposes. Enough is enough, one would think. However, every textbook—and this one is no exception—is both an expression of dissatisfaction with existing books and a statement by the author of what he thinks such a book ought to contain, and a preface offers one last chance to be heard and understood. Furthermore, anyone who adds to the glut of introductory calculus books should be called upon to justify his action (or perhaps apologize for it) to his colleagues in the mathematics community.

THE CALCULUS TURMOIL

I borrow this phrase from my old friend Paul Halmos as a handy label for the noise and confusion that have agitated the calculus community for the past dozen years or so. Regardless of one's attitude toward these debates and manifestoes, it seems reasonably clear that two opinions lie at the center of it all: first, too many students fail calculus; and second, our calculus textbooks are so bad that it's natural for these students to fail.

About the books, I completely—or almost completely—disagree. By and large, our calculus textbooks are written by excellent teachers who love their subject and write clear expository English. Naturally, each author has a personal agenda, and this is what separates their books from one another and provides diversity and choice for a healthy marketplace. Some writers prefer to emphasize the theoretical parts of calculus. Others are technology buffs. Yet others (like myself) want a modest amount of biography and history, and believe that interesting and substantial applications from other parts of mathematics and other sciences are highly desirable.

But let there be no misunderstanding: textbooks are servants of teachers, and not their masters. Any group of ten calculus teachers gathered together in a room will have ten very different views of what should be in their courses and how it should be taught. They will differ on the proper amount of theory; on how much numerical calculation is desirable; on whether or not to make regular use of graphing calculators or computer software; on whether some of the more elaborate applications to science are too difficult; on whether biography and history are interesting or boring for their students; and so on. But the bottom line is that only the teachers themselves are in a position to decide what goes on in their own classrooms—and certainly not textbook writers who are completely ignorant of local conditions.

Those of us who write these books try to provide everything we can think of that a teacher might want or need, in full awareness that some parts of what we offer have no place in the course plans of many teachers. Every teacher omits some sections (and even some chapters) and amplifies others, in accordance with individual judgment and personal taste. It is my hope that this book will be useful and agrèeable for many diverse tastes and interests. I want it to be a convenient tool for teachers that offers help when help is wanted, and gets out of the way when it is not wanted.

As for the fact that too many of our students fail—if indeed it is a fact—what are the reasons for this? To understand these reasons, let us consider for a moment what is needed for success in calculus. There are clearly three main requirements: a decent background in high school algebra and geometry, some of which is remembered and understood; the ability to read closely and carefully; and tenacity of purpose.

In the matter of preparation in algebra and geometry, our students are in deep trouble. This is suggested by the fact that a few years ago the United States ranked last among the thirteen industrialized nations for the mathematics achievement of its high school graduates. As for reading skills and tenacity of purpose, some of our young people have these qualities, but the great majority do not. Unfortunately, tenacity of purpose is especially important for genuine success in calculus, because this is a subject in which almost every stage depends on having a reasonable command of all that went before, and which therefore requires steady application day after day, week after week, for many months.

We know from our own experience as teachers that calculus is very difficult for most students, and we fully understand the reasons why this is so. But improving our high school mathematics education, and arresting the decline of serious reading and instilling tenacity of purpose among the majority of our young people, are only remote possibilities. Obviously help from outside is not coming, so we must look within ourselves for better ways of doing our jobs.

Most of these ways are familiar to us. Regular class meetings over periods of many months, with frequent quizzes, are intended to encourage steady application to the task of learning. We praise (whenever possible), plead, cajole, and warn. We constantly review the elementary mathematics our students either never learned or have forgotten. We do today's homework problems for them in class, continually thinking out loud and welcoming questions, in the hope that some of the useful ways of thought will rub off to smooth the path for their efforts on tomorrow's homework. However, there is one big thing we can do but rarely do.

Most calculus courses concentrate on the technical details, on developing in students the ability to differentiate and integrate lots of functions. We turn out many students who can perform these somewhat routine tasks. However, if we regularly pause to ask these successful differentiators and integrators just what derivatives and integrals actually are, and what they are for, we rarely get a satisfactory answer—by which I mean an answer that reveals genuine understanding on the part of the student. Many can give the standard limit definitions, but we should expect more than parroted formal definitions. I believe we ought to do a better job of conveying a solid sense of what calculus is really about, what its purpose is, why we need the elaborate machinery of methods for computing derivatives and integrals, and why the Fundamental Theorem of Calculus is truly "fundamental." In a word, we need to communicate what calculus is *for.* More generally, we ought to do more toward encouraging students to learn *why* things are true, rather than merely memorizing ways of solving a few problems to pass examinations. It is clear to us, but not to them, that the only way to learn calculus is to understand it—it is much too massive and complex for mere memorizing to be more than a temporary stopgap—and we have an obligation to help students get this message.

If we can give more attention to these matters, we have a good chance of making calculus less frightening and more relevant for many more students than we have in the past. One of the main purposes of this book is to help us move our teaching in this direction, to convey more light to our students—and less mystery.

CHANGES FROM THE FIRST EDITION

1. **Early Trig.** In the First Edition, I thought it preferable to place trigonometry just before methods of integration. I still agree with myself, but most users think otherwise. I have therefore inserted an account of sines and cosines in Chapter 1, with the calculus of these functions at appropriate places in the following chapters. Since a solid command of trigonometry is so essential for methods of integration, a full review is still given just before the chapter on these methods (Chapter 10).

2. **Homework Problems.** I have added many new problems, mostly of the routine drill type, raising the total to well over 7,000. This is an increase of more than 15 percent and provides about four times as many as most instructors will want to use for their class assignments.

3. **Chapter Summaries.** It seems to help students in their efforts to review and pull things together if they have the ideas and methods of each chapter boiled down to a few pregnant phrases. I have tried to provide this assistance in the summaries at the ends of the chapters.

4. **Appendices.** The first edition had several massive appendices totaling hundreds of pages and containing enrichment material that I thought was so interesting that others would be interested, too. Many were, but I failed to realize that students barely keeping their heads above water in the regular work of the course would take a dim view of any unnecessary burdens. The first two of these long appendices were a collection of material that I thought of as "miscellaneous fun stuff," and a biographical history of calculus. These have been removed, augmented, and published separately in a little paperback book called *Calculus Gems: Brief Lives and Memorable Mathematics* (McGraw-Hill, 1992). However, I have retained some of this material in greatly abbreviated form and placed it in unobtrusive locations throughout the present book.

5. **Theory.** The third of the long appendices in the first edition was on the theory of calculus. I have retained this appendix with a few additions because many colleges and universities offer honors sections that use this material to provide greater theoretical depth than is appropriate for regular sections. Most instructors seem to agree with me in my desire to avoid cluttering our regular courses with any more theory than is absolutely necessary. This approach says: Do not try to prove what no one doubts. However, a number of people have asked me to expand my very condensed discussion of limits and continuous functions and also to give an informal descriptive treatment of the Mean Value Theorem, pointing out its practical uses as they arise. This new material can be found at the end of Chapter 2.

6. **Infinite Series.** My idea for handling this subject in the first edition was not a good one. Most students moving from the first chapter of informal overview into the second of detailed systematic treatment were impatient because they thought they were wasting their time by studying the same concepts all over again. I have therefore completely reorganized these two chapters into a traditional treatment, with series of constants developed first, and then power series.

7. **Vector Analysis.** In the first edition I closed my discussion of vector analysis with Green's Theorem. However, there seems to be general agreement these days that multivariable calculus should go a bit further, and include Gauss's Theorem (the divergence theorem) and Stokes' Theorem. I have rewritten Chapter 21 accordingly.

8. **The Workman Logo.** I thought it would be useful for students if there were some way to signal passages in the text that always cause trouble, because most students are not accustomed to the very slow and careful reading these passages require. The logo I chose for this purpose is copied from a European road sign:

It suggests that hard work is necessary to get through the adjoining passage. I have tried to use it sparingly.

9. **Simplify, Simplify!** When writing this book the first time, I thought I was aiming at the middle of my target, but many users thought I aimed too high. During the preparation of this revision, I kept a poster with these words on it directly in my line of sight as I sat at my work, and of course I looked at this message thousands of times. I hope it worked.

GRAPHING CALCULATORS

These marvelous tools are great fun to use and can make many contributions to the teaching and learning of calculus. But like all tools they should be used wisely, and this means very different things to different people. A scythe can harvest grain or cut off a foot, depending on the skill and judgment of the user.

Some of those in the calculus reform movement believe that the role of numbers and numerical computations should be greatly increased to reach a parity with symbolic (algebraic) and geometric ways of thinking. But I believe we should stop far short of this. In my opinion, there are five subject areas of calculus in which calculators are clearly of great value:

graphing;
calculation of limits;
Newton's method;
numerical integration;
computations using Taylor's formula.

In the last four of these areas, our calculators do heavy computational labor for us, and we are all grateful. But there are dangers, and one of these is an increasing tendency to replace mathematical thinking and learning by button-pushing.

The most surprising examples of this that I've seen involve teachers whose students use graphing calculators—*instead of* factoring or the quadratic formula—to solve quadratic equations as simple as $x^2 - 2x - 3 = 0$. The procedure is to "plot" the function $y = x^2 - 2x - 3$ on the calculator by pushing suitable buttons and then look at the graph the calculator produces to see where it crosses the x-axis. These students are enthusiastic about their calculators and enjoy experimenting with them, and I applaud the teachers who take advantage of this natural interest. But unfortunately, in many cases these students *do not know* how to sketch simple graphs, or how to factor or use the quadratic formula, and are not learning these basic methods of elementary algebra. More generally, sketching the graphs of functions by *thinking* is a fundamental part of learning mathematics. Let us use calculators in our classes to supplement this thinking—but not to replace it. Let us remember that the action that matters takes place in the mind of the student.

These wonderful graphing calculators are superb instruments when used in the right way. It is sobering to reflect that Leibniz himself would perhaps have given a year of his life to possess one—Leibniz who not only (along with Newton) created calculus, but also invented the first calculating machine that could multiply and divide as well as add and subtract.

The many problems in this book that require the use of a calculator are signaled by the standard symbol .

THE PURPOSE OF THIS BOOK

This book is intended to be a mainstream calculus text that is suitable for every kind of course at every level. It is designed particularly for the standard course of three semesters for students of science, engineering, or mathematics. Students are expected to have a background of high school algebra and geometry, and hopefully, some trigonometry as well.

The text itself—that is, the 21 chapters without considering Appendix A—is traditional in subject matter and organization. I have placed great emphasis on *motivation* and *intuitive understanding,* and the refinements of theory are downplayed. Most students are impatient with the theory of the subject, and justifiably so, because the essence of calculus does not lie in theorems and how to prove them, but rather in tools and how to use them. My overriding purpose has been to present calculus as a problem-solving art of immense power that is indispensable in all the quantitative sciences. Naturally, I wish to convince students that the standard tools of calculus are reasonable and legitimate, but not at the expense of turning the subject into a stuffy logical discipline dominated by extra-careful definitions, formal statements of theorems, and meticulous proofs. It is my hope that every mathematical explanation in these chapters will seem to the thoughtful student to be as natural and inevitable as the fact that water flows downhill (rather than uphill) along a canyon floor. The main theme of our work is what calculus is good for—what it enables us to do and understand—and not what its logical nature is as seen from the specialized (and limited) point of view of the modern pure mathematician.

There are several additional features of the book that it might be useful for me to comment on.

Precalculus Material Because of the great amount of calculus that must be covered, it is desirable to get off to a fast start and introduce the derivative quickly, and to spend as little time as possible reviewing precalculus material. However, college freshmen constitute a very diverse group, with widely different levels of mathematical preparation. For this reason I have included a first chapter on precalculus material, which I urge teachers to skim over as lightly as they think advisable for their particular students. This chapter is written in enough detail so that individual students who need to spend more time on the preliminaries should be able to absorb most of it on their own with a little extra effort.*

Problems For students, the most important parts of their calculus book may well be the problem sets, because this is where they spend most of their time and energy. There are more than 7,000 problems in this book, including many old standbys familiar to all calculus teachers and dating back to the time of Euler and even earlier. I have tried to repay my debt to the past by inventing new problems whenever possible. The problem sets are carefully constructed, beginning with routine drill exercises and building up to more complex problems requiring higher levels of thought and skill. The most challenging problems are marked with an asterisk (*). In general, each set contains approximately twice as many problems as most teachers will want to assign for homework, so that a large number will be left over for students to use as review material.

Most of the chapters conclude with long lists of additional problems. Many of these are intended only to provide further scope and variety to the problems sets at the ends of the sections. However, teachers and students alike should treat these additional problems with special care, because a few are quite subtle and difficult and should be attacked only by students with ample reserves of drive and tenacity.

I should also mention that there are several sections scattered throughout the book with no corresponding problems at all. Sometimes these sections occur in small groups and are merely convenient subdivisions of what I consider a single topic and intend as a single assignment, as with Sections 6.1, 6.2, 6.3, and 6.4, 6.5. In other cases (e.g., Sections 15.5 and 19.4), the absence of problems is a tacit suggestion that the subject matter of these sections should be touched upon only lightly and briefly.

There are a great many so-called "story problems" spread through the entire book. All teachers know that students shudder at these problems, because they usually require nonroutine thinking. However, the usefulness of mathematics in the various sciences demands that we try to teach our students how to penetrate into the meaning of a story problem, how to judge what is relevant to it, and how to translate it from words into sketches and equations. Without these skills—which are equally valuable for students who will become doctors, lawyers, financial analysts, or thinkers of any kind—there is no mathematics education worthy of the name.†

THE FAR SIDE By GARY LARSON

Hell's library

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*A more complete exposition of high school mathematics that is still respectably concise can be found in my little book, *Precalculus Mathematics in a Nutshell* (Janson Publications, Dedham, MA, 1981), 119 pages.

†I cannot let the opportunity pass without quoting a classic story problem that appeared in *The New Yorker* magazine many years ago. "You know those terrible arithmetic problems about how many peaches some people buy, and so forth? Well, here's one we *like,* made up by a third-grader who was asked to think up a problem similar to the ones in his book: "My father is forty-four years old. My dog is eight. If my dog was a human being, he would be fifty-six years old. How old would my father plus my dog be if they were both human beings?"

Differential Equations and Vector Analysis Each of these subjects is an important branch of mathematics in its own right. They should be taught in separate courses, after calculus, with ample time to explore their distinctive methods and applications. One of the main responsibilities of a calculus course is to prepare the way for these more advanced subjects and take a few preliminary steps in their direction, but just how far one should go is a debatable question. Some writers on calculus try to include mini-courses on these subjects in large chapters at the ends of their books. I disagree with this practice and believe that few teachers make much use of these chapters. Instead, in the case of differential equations I prefer to introduce the subject as early as possible (Section 5.4) and return to it in a low-key way whenever the opportunity arises (Sections 5.5, 7.7, 8.5, 9.6, 17.7, 19.9); and in vector analysis I have responded to reviewers by including a discussion of Gauss's Theorem and Stokes' Theorem in Chapter 21.

Appendix A One of the major ways in which this book is unique and different from all its competitors can be understood by examining Appendix A, which I will now comment on very briefly. Before doing so, I emphasize that this material is entirely separate from the main text and can be carefully studied, dipped into occasionally, or completely ignored, as each individual student or instructor desires.

In the main text, the level of mathematical rigor rises and falls in accordance with the nature of the subject under discussion. It is rather low in the geometrical chapters, where for the most part I rely on common sense together with intuition aided by illustrations; and it is rather high in the chapters on infinite series, where the substance of the subject cannot really be understood without careful thought. I have constantly kept in mind the fact that most students have very little interest in purely mathematical reasoning for its own sake, and I have tried to prevent this type of material from intruding any more than is absolutely necessary. Some students, however, have a natural taste for theory, and some instructors feel as a matter of principle that all students should be exposed to a certain amount of theory for the good of their souls. This appendix contains virtually all of the theoretical material that by any stretch of the imagination might be considered appropriate for the study of calculus. From the purely mathematical point of view, it is possible for instructors to teach courses at many different levels of sophistication by using—or not using—material selected from this appendix.

Supplements The following supplements have been developed to accompany this Second Edition of *Calculus with Analytic Geometry*.

A *Student Solutions Manual* is available for students and contains detailed solutions to the odd-numbered problems. An *Instructor's Solutions Manual* is available for instructors and contains detailed solutions to the even-numbered problems. Also available to instructors adopting the text are a Print Test Bank and an algorithmic Computerized Test Bank.

There are a variety of texts available from McGraw-Hill that support the use of specific graphing calculators and mathematical software programs for calculus. Please contact your local McGraw-Hill representative for more information on these titles.

ACKNOWLEDGMENTS

Every project of this magnitude obviously depends on the cooperative efforts of many people.

For this second edition, the editor Jack Shira provided friendly encouragement and smoothed my way throughout. I am profoundly grateful to my friend Maggie Lanzillo, the associate editor, who was a source of skilled support, assistance, and guidance on innumerable occasions—extending even to restaurant suggestions for dining in Italy. Thanks, Maggie. I owe you more than I can express. And as another piece of extraordinary good luck, this second edition was designed by Joan O'Connor, who designed the first edition, and whose inspired artistic taste seems to work miracles on a daily basis.

Also, I offer my sincere thanks to the publisher's reviewers. These astute people shared their knowledge and judgment with me in many important ways.

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As to the flaws and errors that undoubtedly remain—for there are always a pesky few that manage to hide no matter how fervently we try to find them—there is no one to blame but myself. I will consider it a great kindness if colleagues and student users will take the trouble to inform me of any blemishes they detect, for correction in future printings and editions. As Confucius said, "A man who makes a mistake and doesn't correct it is making two mistakes."

George F. Simmons

TO THE STUDENT

Appearances to the contrary, no writer deliberately sets out to produce an unreadable book; we all do what we can and hope for the best. Naturally, I hope that my language will be clear and helpful to students, and in the end only they are qualified to judge. However, it would be a great advantage to all of us—teachers and students alike—if student users of mathematics textbooks could somehow be given a few hints on the art of reading mathematics, which is a very different thing from reading novels or magazines or newspapers.

In high school mathematics courses, most students are accustomed to tackling their homework problems first, out of impatience to have the whole burdensome task over and done with as soon as possible. These students read the explanations in the text only as a last resort, if at all. This is a grotesque reversal of reasonable procedure, and makes about as much sense as trying to put on one's shoes before one's socks. I suggest that students should read the text first, and when this has been thoroughly assimilated, *then and only then* turn to the homework problems. After all, the purpose of these problems is to nail down the ideas and methods described and illustrated in the text.

How should a student read the text in a book like this? Slowly and carefully, and in full awareness that a great many details have been deliberately omitted. If this book contained every detail of every discussion, it would be five times as long, which God forbid! There is a saying of Voltaire: "The secret of being a bore is to tell everything." Every writer of a book of this kind tries to walk a narrow path between saying too much and saying too little.

The words "clearly," "it is easy to see," and similar expressions are not intended to be taken literally, and should never be interpreted by any student as a putdown on his or her abilities. These are code-phrases that have been used in mathematical writing for hundreds of years. Their purpose is to give a signal to the careful reader that in this particular place, the exposition is somewhat condensed, and perhaps a few details of calculations have been omitted. Any phrase like this amounts to a friendly hint to the student that it might be a good idea to read even more carefully and thoughtfully in order to fill in omissions in the exposition, or perhaps get out a piece of scratch paper to verify omitted details of calculations. Or better yet, make full use of the margins of this book to emphasize points, raise questions, perform little computations, and correct misprints.

George F. Simmons

本书除具有标准微积分教材的内容外，书中例子偏重实际，侧重于微积分的应用，同时补充了三角函数、极坐标等理论知识，使学生从高中到大学平稳过渡。文中穿插数学史与数学文化的相关内容，同时附录中提供了大量的补充内容以及严格的理论证明，适合不同层次的学生按需要学习。附加问题生动有趣，多是相关内容的经典结论。

本书可作为高等院校理工科专业教材，也可作为相关科研、技术人员的参考书。

George F. Simmons

Calculus With Analytic Geometry

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目　　录

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PART III

1 NUMBERS, FUNCTIONS, AND GRAPHS

1.1 INTRODUCTION

Everyone knows that the world in which we live is dominated by motion and change. The earth moves in its orbit around the sun; a rock thrown upward slows and stops, and then falls back to earth with increasing speed; the population of India grows each year at an increasing rate; and radioactive elements decay. These are merely a few items in the endless array of phenomena for which mathematics is the most natural medium of communication and understanding. As Galileo said more than 300 years ago, "The Great Book of Nature is written in mathematical symbols."

Calculus is that branch of mathematics whose primary purpose is the study of motion and change. It is an indispensable tool of thought in almost every field of pure and applied science—in physics, chemistry, biology, astronomy, geology, engineering, and even some of the social sciences. It also has many important uses in other parts of mathematics, especially geometry. By any standard, the methods and applications of calculus constitute one of the greatest intellectual achievements of civilization, and to become acquainted with these ideas is to open many doors that lead to a broader and richer life of the mind.

The main objects of study in calculus are functions. But what is a function? Roughly speaking, it is a rule or law that tells us how one variable quantity depends upon another. This is the master concept of the exact sciences. It offers us the prospect of understanding and correlating natural phenomena by means of mathematical machinery of great and sometimes mysterious power. The concept of a function is so vitally important for all our work that we must strive to clarify it beyond any possibility of confusion. This purpose is the theme of the present chapter.

The following sections contain a good deal of material that many readers have studied before. Some will welcome the opportunity to review and refresh their ideas. Those who already understand this material and find it irksome to tread the same path over again may discover some interesting sidelights and stimulating challenges among the Additional Problems at the end of the chapter. This chapter is intended solely for purposes of review. It can be studied carefully, or lightly, or even skipped altogether, depending on the reader's level of preparation. The actual subject matter of this course begins in Chapter 2, and it would be very unfortunate if even a single student should come to feel that this preliminary chapter is more of an obstacle than a source of assistance—for its only purpose is to smooth the way.

1.2 THE REAL LINE AND COORDINATE PLANE. PYTHAGORAS

Most of the variable quantities we study—such as length, area, volume, position, time, and velocity—are measured by means of real numbers, and in this sense calculus is based on the real number system. It is true that there are other important and useful number systems—for instance, the complex numbers. It is also true that two- and three-dimensional treatments of position and velocity require the use of vectors. These ideas will be examined in due course, but for a long time to come the only numbers we shall be working with are the real numbers.*

It is assumed in this book that students are familiar with the elementary algebra of the real number system. Nevertheless, in this section we give a brief descriptive survey that may be helpful. For our purposes this is sufficient, but any reader who wishes to probe more deeply into the nature of real numbers will find a more precise discussion in Appendix A.1 at the back of the book.

The real number system contains several types of numbers that deserve special mention: the *positive integers* (or *natural numbers*)

$$1, 2, 3, 4, 5, \ldots ;$$

the *integers*

$$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots ;$$

and the *rational numbers,* which are those real numbers that can be represented as fractions (or quotients of integers), such as

$$\tfrac{2}{3}, -\tfrac{7}{4}, 4, 0, -5, 3.87, 2\tfrac{1}{4}.$$

A real number that is not rational is said to be *irrational;* for example,

$$\sqrt{2}, \sqrt{3}, \sqrt{2}+\sqrt{3}, \sqrt{5}, \sqrt[3]{5}, \qquad \text{and} \qquad \pi$$

are irrational numbers.†

We take this opportunity to remind the reader that for any positive number a, the symbol $\sqrt{a}$ always means its positive square root. Thus, $\sqrt{4}$ is equal to 2 and not -2, even though $(-2)^2 = 4$. If we wish to designate both square roots of 4, we must write $\pm\sqrt{4}$. Similarly, $\sqrt[n]{a}$ always means the positive nth root of a.

THE REAL LINE

The use of the real numbers for measurement is reflected in the very convenient custom of representing these numbers graphically by points on a horizontal straight line (Fig. 1.1).

This representation begins with the choice of an arbitrary point as the origin or zero point, and another arbitrary point to the right of it as the point 1. The dis-

*The adjective "real" was originally used to distinguish these numbers from numbers like $\sqrt{-1}$, which were once thought to be "unreal" or "imaginary."

†Our aims in the present book are almost entirely practical. Nevertheless, our discussions often give rise to certain "impractical" questions that some readers may find interesting and appealing. As an example, how do we know that the number $\sqrt{2}$ is irrational? For readers with the time and inclination to pursue such questions—and also because we consider the answers to be worth knowing about for their own sake—we offer food for further thought in a little paperback book entitled *Calculus Gems: Brief Lives and Memorable Mathematics* (McGraw-Hill, 1992). Some of the facts about irrational numbers, with proofs, are discussed in Section B.2 of this book.

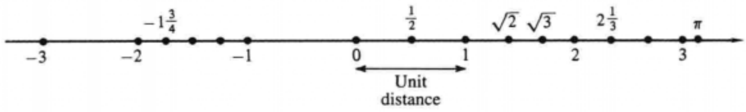

Figure 1.1 The real line.

tance between these two points (the unit distance) then serves as a scale by means of which we can assign a point on the line to every positive and negative integer, as illustrated in the figure, and also to every rational number. Notice that all positive numbers lie to the right of 0, in the "positive direction," and all negative numbers lie to the left.* The method of assigning a point to a rational number is shown in the figure for the number $\frac{7}{3} = 2\frac{1}{3}$: the segment between 2 and 3 is subdivided by two points into three equal segments, and the first of these points is labeled $2\frac{1}{3}$. This procedure of using equal subdivisions clearly serves to determine the point on the line which corresponds to any rational number whatever. Furthermore, this correspondence between rational numbers and points can be extended to irrational numbers; for the decimal expansion of an irrational number, such as

$$\sqrt{2} = 1.414\ldots, \qquad \sqrt{3} = 1.732\ldots, \qquad \pi = 3.14159\ldots,$$

can be interpreted as a set of instructions specifying the exact position of the corresponding point. For example, by looking at the expansion we see that the point corresponding to $\sqrt{2}$ lies between 1 and 2, between 1.4 and 1.5, between 1.41 and 1.42, and so on, and these requirements uniquely determine the position of the corresponding point.

We have described a one-to-one correspondence between all real numbers and all points on the line which establishes these numbers as a coordinate system for the line. This coordinatized line is called the *real line.* It is convenient and customary to merge the logically distinct concepts of the real number system and the real line, and we shall freely speak of points on the line as if they were numbers and of numbers as if they were points on the line. Thus, such mixed expressions as "irrational point" and "the segment between 2 and 3" are quite natural and will be used without further comment.

INEQUALITIES

The left-to-right linear succession of points on the real line corresponds to an important part of the algebra of the real number system, that dealing with inequalities. These ideas play a larger role in calculus than in earlier mathematics courses, so we briefly recall the essential points.

The geometric meaning of the inequality $a < b$ (read "a is less than b") is simply that a lies to the left of b; the equivalent inequality $b > a$ ("b is greater than a") means that b lies to the right of a. A number a is positive or negative according as $a > 0$ or $a < 0$. The main rules used in working with inequalities are the following:

*The arrowhead on the right end of the real line indicates the positive direction and nothing more.

1. If $a > 0$ and $b < c$, then $ab < ac$.
2. If $a < 0$ and $b < c$, then $ab > ac$.
3. If $a < b$, then $a + c < b + c$ for any number c.

Rules 1 and 2 are usually expressed by saying that an inequality is preserved on multiplication by a positive number, and reversed on multiplication by a negative number; and rule 3 says that an inequality is preserved when any number (positive or negative) is added to both sides. It is often desirable to replace an inequality $a > b$ by the equivalent inequality $a - b > 0$, with rule 3 being used to establish the equivalence.

If we wish to say that a is positive or equal to 0, we write $a \geq 0$ and read this "a is greater than or equal to zero." Similarly, $a \geq b$ means that $a > b$ or $a = b$. Thus, $3 \geq 2$ and $3 \geq 3$ are both true inequalities.

We also recall that a product of two or more numbers is zero if and only if one of its factors is zero. If none of its factors are zero, it is positive or negative according as it has an even or an odd number of negative factors.

ABSOLUTE VALUES

The *absolute value* of a number a is denoted by $|a|$ and defined by

$$|a| = \begin{cases} a & \text{if } a \geq 0, \\ -a & \text{if } a < 0. \end{cases}$$

For example, $|3| = 3$, $|-2| = -(-2) = 2$, and $|0| = 0$. It is clear that the operation of forming the absolute value leaves positive numbers unchanged and replaces each negative number by the corresponding positive number. The main properties of this operation are

$$|ab| = |a||b| \qquad \text{and} \qquad |a + b| \leq |a| + |b|.$$

In geometric language, the absolute value of a number a is simply the distance from the point a to the origin. Similarly, the distance from a to b is $|a - b|$.

To solve an equation such as $|x + 2| = 3$, we can write it in the form $|x - (-2)| = 3$ and think of it as saying that "the distance from x to -2 is 3." With Fig. 1.1 in mind, it is evident that the solutions are $x = 1$ and $x = -5$. We can also solve this equation by using the fact that $|x + 2| = 3$ means that $x + 2 = 3$ or $x + 2 = -3$; the solutions are $x = 1$ and $x = -5$, as before.

INTERVALS

The sets of real numbers we shall be dealing with most frequently are intervals. An *interval* is simply a segment on the real line. If its endpoints are the numbers a and b, then the interval consists of all numbers that lie between a and b. However, we may or may not want to include the endpoints themselves as part of the interval.

To be more precise, suppose that a and b are numbers, with $a < b$. The *closed interval* from a to b, denoted by $[a, b]$—using brackets—includes its endpoints, and therefore consists of all real numbers x such that $a \leq x \leq b$. Parentheses are used to indicate excluded endpoints. The interval (a, b), with both endpoints excluded, is called the *open interval* from a to b; it consists of all x such that

$a < x < b$. Sometimes we wish to include only one endpoint in an interval. Thus, the intervals denoted by $[a, b)$ and $(a, b]$ are defined by the inequalities $a \le x < b$ and $a < x \le b$, respectively. In each of these cases, any number c such that $a < c < b$ is called an *interior point* of the interval (Fig. 1.2).

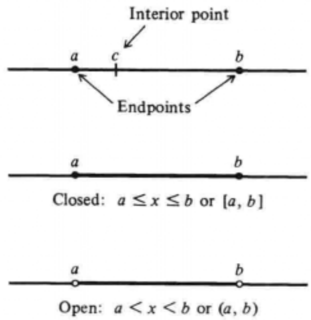

Figure 1.2 Intervals.

Strictly speaking, the notations $a \le x \le b$ and $[a, b]$ have different meanings —the first represents a restriction imposed on x, while the second denotes a set —but both designate the same interval. We will therefore consider them to be equivalent and use them interchangeably, and the reader should become familiar with both. However, the geometric meaning of the notation $a \le x \le b$ is more easily grasped by the eye, and for this reason we usually prefer it to the other.

A half-line is often considered to be an interval extending to infinity in one direction. The symbol ∞ (read "infinity") is frequently used in designating such an interval. Thus, for any real number a the intervals defined by the inequalities $a < x$ and $x \le a$ can be written as $a < x < \infty$ and $-\infty < x \le a$, or equivalently as (a, ∞) and $(-\infty, a]$. Remember, however, that the symbols ∞ and $-\infty$ do not denote real numbers; they are used in this manner only as a convenient way of emphasizing that x is allowed to be arbitrarily large (in either the positive or negative direction). As an aid in keeping the notation clear in one's mind, it may be helpful to think of $-\infty$ and ∞ as "fictitious numbers" located at the left and right "ends" of the real line, as suggested in Fig. 1.3. Also, it is sometimes convenient to think of the entire real line itself as an interval, $-\infty < x < \infty$ or $(-\infty, \infty)$.

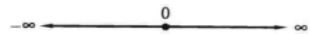

Figure 1.3

Sets of numbers described by means of inequalities and absolute values are often intervals. It is clear, for instance, that the set of all x such that $|x| < 2$ is the interval $-2 < x < 2$ or $(-2, 2)$.

Example 1 Solve the inequality $x^2 - 2 < x$.

Solution To "solve" an inequality like this means to find all numbers x for which the inequality is true. We begin by writing it as $x^2 - x - 2 < 0$, and then we write it in the factored form

$$(x + 1)(x - 2) < 0.$$

For this to be true, the two factors must have opposite signs: $x + 1 > 0$ and $x - 2 < 0$, or $x + 1 < 0$ and $x - 2 > 0$. These conditions are equivalent to $x > -1$ and $x < 2$, or $x < -1$ and $x > 2$. The second pair of conditions is easily seen to be impossible. The first pair of conditions means that x lies in the open interval $-1 < x < 2$, and these x's constitute the solution of the given inequality.

THE COORDINATE PLANE

Just as real numbers are used as coordinates for points on a line, pairs of real numbers can be used as coordinates for points in a plane. For this purpose we establish a *rectangular coordinate system* in the plane, as follows.

Draw two perpendicular straight lines in the plane, one horizontal and the other vertical, as shown in Fig. 1.4. These lines are called the *x-axis* and *y-axis*, respectively, and their point of intersection is called the *origin*. Coordinates are assigned to these axes in the manner described earlier, with the origin as the zero point on both and the same unit of distance measurement on both. The positive

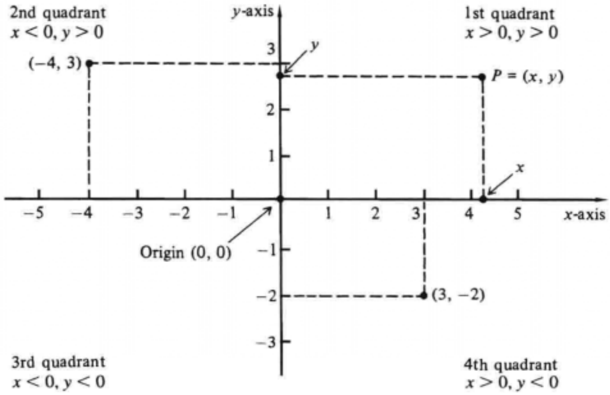

Figure 1.4 The coordinate plane or xy-plane.

x-axis is to the right of the origin and the negative x-axis to the left, as before; and the positive y-axis is above the origin and the negative y-axis below.

Now consider a point P anywhere in the plane. Draw a line through P parallel to the y-axis, and let x be the coordinate of the point where this line crosses the x-axis. Similarly, draw a line through P parallel to the x-axis, and let y be the coordinate of the point where this line crosses the y-axis. The numbers x and y determined in this way are called the *x-coordinate* and *y-coordinate* of P. In referring to the coordinates of P, it is customary to write them as an ordered pair (x, y) with the x-coordinate written first; we say that P has coordinates (x, y).* This correspondence between P and its coordinates establishes a one-to-one correspondence between all points in the plane and all ordered pairs of real numbers; for P determines its coordinates uniquely, and by reversing the process we see that each ordered pair of real numbers uniquely determines a point P with these numbers as its coordinates. As in the case of the real line, it is customary to drop the distinction between a point and its coordinates, and to speak of "the point (x, y)" instead of "the point with coordinates (x, y)." The coordinates x and y of the point P are sometimes called the *abscissa* and *ordinate* of P. Notice particularly that points $(x, 0)$ lie on the x-axis, that points $(0, y)$ lie on the y-axis, and that $(0, 0)$ is the origin. Also, the axes divide the plane into four quadrants, as shown in Fig. 1.4, and these quadrants are characterized as follows by the signs of x and y: first quadrant, $x > 0$ and $y > 0$; second quadrant, $x < 0$ and $y > 0$; third quadrant, $x < 0$ and $y < 0$; fourth quadrant, $x > 0$ and $y < 0$.

When the plane is equipped with the coordinate system described here, it is usually called the *coordinate plane* or the *xy-plane.*

THE DISTANCE FORMULA

Much of our work involves geometric ideas—right triangles, similar triangles, circles, spheres, cones, etc.—and we assume that students have acquired a reasonable grasp of elementary geometry from earlier mathematics courses. A ma-

*In practice, the use of the same notation for ordered pairs as for open intervals never leads to confusion, because in any specific context it is always clear which is meant.

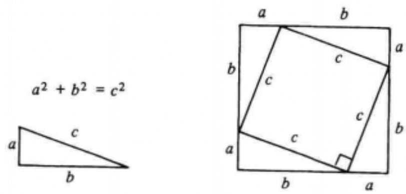

Figure 1.5 The Pythagorean theorem and a proof.

jor fact of particular importance is the Pythagorean theorem: In any right triangle, the sum of the squares of the legs equals the square of the hypotenuse (Fig. 1.5). There are many proofs of this theorem, but the following is probably simpler than most. Let the legs be a and b and the hypotenuse c, and arrange four replicas of the triangle in the corners of a square of side $a + b$, as shown on the right in Fig. 1.5. Then the area of the large square equals 4 times the area of the triangle plus the area of the small square; that is,

$$(a + b)^2 = 4(\tfrac{1}{2}ab) + c^2.$$

This simplifies at once to $a^2 + b^2 = c^2$, which is the Pythagorean theorem.

As the first of many applications of this fact, we obtain the formula for the distance d between any two points in the coordinate plane. If the points are $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, then the segment joining them is the hypotenuse of a right triangle (Fig. 1.6) with legs $|x_1 - x_2|$ and $|y_1 - y_2|$. By the Pythagorean theorem,

$$\begin{aligned} d^2 &= |x_1 - x_2|^2 + |y_1 - y_2|^2 \\ &= (x_1 - x_2)^2 + (y_1 - y_2)^2, \end{aligned}$$

so

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}. \tag{1}$$

This is the *distance formula.*

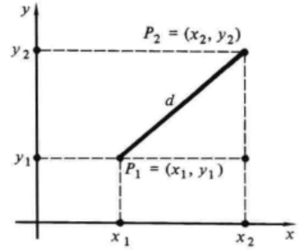

Figure 1.6

Example 2 The distance d between the points $(-4, 3)$ and $(3, -2)$ in Fig. 1.4 is

$$d = \sqrt{(-4 - 3)^2 + (3 + 2)^2} = \sqrt{74}.$$

Notice that in applying formula (1) it does not matter in which order the points are taken.

Example 3 Find the lengths of the sides of the triangle whose vertices are $P_1 = (-1, -3)$, $P_2 = (5, -1)$, $P_3 = (-2, 10)$.

By (1), these lengths are

$$\begin{aligned} P_1P_2 &= \sqrt{(-1 - 5)^2 + (-3 + 1)^2} = \sqrt{40} = 2\sqrt{10}, \\ P_1P_3 &= \sqrt{(-1 + 2)^2 + (-3 - 10)^2} = \sqrt{170}, \\ P_2P_3 &= \sqrt{(5 + 2)^2 + (-1 - 10)^2} = \sqrt{170}. \end{aligned}$$

These calculations reveal that the triangle is isosceles, with P_1P_3 and P_2P_3 as the equal sides.

THE MIDPOINT FORMULAS

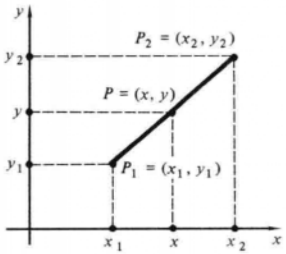

Figure 1.7

It is often useful to know the coordinates of the midpoint of the segment joining two given distinct points. If the given points are $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, and if $P = (x, y)$ is the midpoint, then it is clear from Fig. 1.7 that x is the midpoint of the projection of the segment on the x-axis, and similarly for y. This tells us (examine the figure—and think!) that $x = x_1 + \frac{1}{2}(x_2 - x_1)$ and $y = y_1 + \frac{1}{2}(y_2 - y_1)$, so

$$x = \tfrac{1}{2}(x_1 + x_2) \qquad \text{and} \qquad y = \tfrac{1}{2}(y_1 + y_2).$$

Another way of obtaining these formulas is to notice from Fig. 1.7 that $x - x_1 = x_2 - x$, so $2x = x_1 + x_2$ or $x = \frac{1}{2}(x_1 + x_2)$, with the same argument applying to y. Similarly, if P is a trisection point of the segment joining P_1 and P_2, its coordinates can be found from the fact that x and y are trisection points of the corresponding segments on the x-axis and y-axis.

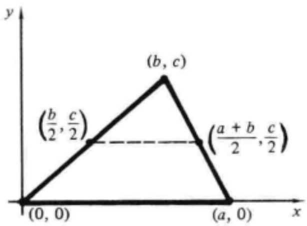

Figure 1.8

Example 4 In any triangle, the segment joining the midpoints of two sides is parallel to the third side and half its length. We know this from elementary geometry; but to prove it by our methods, we begin by noticing that the triangle can always be placed in the position shown in Fig. 1.8, with its third side along the positive x-axis and the left endpoint of this side at the origin. We then insert the midpoints of the other two sides, as shown, and observe that since they have the same y-coordinate, the segment joining them is parallel to the third side lying on the x-axis. The length of this segment is simply the difference between the x-coordinates of its endpoints,

$$\frac{a+b}{2} - \frac{b}{2} = \frac{a}{2},$$

which is half the length of the third side.

This example illustrates the way in which coordinates can often be used to give algebraic proofs of geometric theorems. The device employed here, of placing the figure in a convenient position relative to the coordinate system, has the purpose of simplifying the algebra.

NOTE ON PYTHAGORAS

Who was this Pythagoras, whose name is attached to the great theorem of geometry we have just been using? And why should we care?

The pre-Socratic philosophers of ancient Greece—that is, those who lived before the time of Socrates (470?–399 B.C.)—were one of the most remarkable and influential groups of people in human history. The best known of these was Pythagoras of Samos (580?–500? B.C.), a mathematician, scientist, and mystic whose ideas live on today as part of the bone and flesh of our modern civilization.

Greek geometry was certainly one of the half-dozen supreme intellectual achievements of all time. Pythagoras' master Thales (625?–547? B.C.) had created geometry as the contemplation of abstract patterns of lines and figures and constructed the first proofs of the first theorems. But Pythagoras was the first person to see geometry as an organized system of thought held together by deductive proof, with one theorem depending on another in a tightly woven fabric of logic. Also, tradition tells us that he himself discovered many theorems, most notably, the fact that the sum of the angles in any triangle equals two right angles, and the famous Pythagorean theorem discussed above.

Pythagoras was born on the beautiful island of Samos, a mile or two off the Aegean coast of Turkey and a good day's walk along the shore from Thales' home town of Miletus. At the age of about 50, he migrated from Samos to the Greek colony of Crotona in southern Italy, where he established the famous Pythagorean school, a quasi-religious society with a solid claim to the honor of being the world's first university. The Pythagoreans were best known for two teachings: the doctrine of transmigration of souls at death from one body into another, and the theory that numbers constitute the true essence of all things. Believers performed rites of purification and followed strict moral and dietary rules (no sex, no meat) to enable their souls to rise to higher levels of spirituality in subsequent lives. Their beliefs also led them to consider the sexes as equal and to treat animals and slaves humanely. For who knows? In a subsequent life one might return as a slave, or one's soul might take up residence in an animal's body, or even—alas!—an insect's.

As a way of achieving purification of the mind, the Pythagoreans studied geometry, arithmetic, music, and astronomy—arithmetic not in the sense of useful computational skills but rather as the abstract theory of numbers. They were particularly fond of the "figurate numbers," which arise by arranging dots or points in regular geometric patterns. For example, there are the square numbers 1, 4, 9, 16, . . . :

As indicated, each square number can be obtained from its predecessor by adding an L-shaped border called a *gnomon*, meaning a carpenter's square. Since the successive gnomons are the successive odd numbers, it is immediately clear from the square arrays that the sum of the first n odd numbers equals n^2:

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

Who would have believed that the common odd numbers and the relatively rare perfect squares are related in such a simple yet remarkable way? The Pythagoreans were fascinated, and rightly so, by the grave and beautiful games that numbers play with each other—games that seemed to them to take place outside of space and time and to be quite independent of the human mind itself.

Further, Pythagoras performed the first deliberate scientific experiment, on the relation between positive whole numbers and the musical notes emitted by a plucked lyre string. Also, he was the first person to conceive the supremely daring conjecture that the world is an ordered, understandable whole, and he applied the word *kosmos*—which previously meant order or harmony—to this whole.

In these and other ways Pythagoras was one of the prime creators of the Western civilization that sustains us all—as fish are sustained by the water in which they swim.

PROBLEMS

1 Among the words "integer," "rational," and "irrational," state the ones that apply to

(a) $-\frac{2}{3}$; (b) 0;
(c) $\frac{45}{9}$; (d) 0.75;
(e) $-\sqrt{49}$; (f) $1/\pi$;
(g) 9.000 . . . ; (h) $3^{1/2}$;
(i) $-\frac{20}{7}$; (j) $\frac{94}{7}$.

2 Every integer is either even or odd. The even integers are those that are divisible by 2, so n is even if and only if it has the form $n = 2k$ for some integer k. The odd integers are those that have the form $n = 2k + 1$ for some integer k.

(a) If n is even, prove that n^2 is also even.
(b) If n is odd, prove that n^2 is also odd.

In Problems 3–12, rewrite the given expression without using the absolute value symbol.

3 $|7 - 18|$. **4** $|7| - |-18|$.
5 $|\pi - 3|$. **6** $|3 - \pi|$.
7 $|x - 5|$ if $x < 5$. **8** $|x - 5|$ if $x > 5$.
9 $|x^2 + 10|$. **10** $|-11| - |-10|$.
11 $|1 - 3x^2|$ if $x \geq 1$. **12** $|\sqrt{10} - 10|$.

13 Solve the following inequalities:
(a) $x(x - 1) > 0$;
(b) $(x - 1)(x + 2) < 0$;

(c) $x^2 + 4x - 21 > 0$;
(d) $2x^2 + x < 3$;
(e) $4x^2 + 10x - 6 < 0$;
(f) $x^2 + 2x + 4 > 0$.

14 Recall that $\sqrt{a}$ is a real number if and only if $a \geq 0$, and find the values of x for which each of the following is a real number:
(a) $\sqrt{4 - x^2}$; (b) $\sqrt{x^2 - 9}$;
(c) $\dfrac{1}{\sqrt{4 - 3x}}$; (d) $\dfrac{1}{\sqrt{x^2 - x - 12}}$.

15 Find the values of x for which each of the following is positive:
(a) $\dfrac{x}{x^2 + 4}$; (b) $\dfrac{x}{x^2 - 4}$;
(c) $\dfrac{x + 1}{x - 3}$; (d) $\dfrac{x^2 - 1}{x^2 - 3x}$.

16 State the values of a for which the following inequalities are valid:
(a) $a \leq a$; (b) $a < a$.

17 If $a \leq b$ and $b \leq a$, what conclusion can be drawn about a and b?

18 (a) If $a < b$ is true, is it also necessarily true that $a \leq b$?
(b) If $a \leq b$ is true, is it also necessarily true that $a < b$?

19 State whether each pair of points lies on a horizontal or a vertical line:
(a) $(-2, -5), (-2, 3)$; (b) $(-2, -5), (7, -5)$;
(c) $(-3, 4), (6, 4)$; (d) $(2, -11), (2, 5)$;
(e) $(2, 2), (-13, 2)$; (f) $(-7, -7), (-7, 7)$;
(g) $(3, 5), (3, -2)$; (h) $(-1, -2), (2, -2)$.

20 Three vertices of a rectangle are $(-1, 2)$, $(3, -5)$, $(-1, -5)$. What is the fourth vertex?

21 Find the distance between each pair of points:
(a) $(1, 2), (6, 7)$; (b) $(2, 5), (-1, 3)$;
(c) $(-7, 3), (1, -2)$; (d) $(a, b), (b, a)$.

22 In Problem 21 find the midpoint of the segment joining each pair of points.

23 Draw a sketch indicating the points (x, y) in the plane for which
(a) $x < 2$;
(b) $-1 < y \leq 2$;
(c) $0 \leq x \leq 1$ and $0 \leq y \leq 1$;
(d) $x = -1$;
(e) $y = 3$;
(f) $x = y$.

24 Use the distance formula to show that the points $(-2, 1)$, $(2, 2)$, and $(10, 4)$ lie on a straight line.

25 Show that the point $(6, 5)$ lies on the perpendicular bisector of the segment joining the points $(-2, 1)$ and $(2, -3)$.

26 Show that the triangle whose vertices are $(3, -3)$, $(-3, 3)$, and $(3\sqrt{3}, 3\sqrt{3})$ is equilateral.

27 The two points $(2, -2)$ and $(-6, 5)$ are the endpoints of a diameter of a circle. Find the center and radius of the circle.

28 Find every point whose distance from each of the two coordinate axes equals its distance from the point $(4, 2)$.

29 Find the point equidistant from the three points $(-9, 0)$, $(6, 3)$, and $(-5, 6)$.

30 If a and b are any two numbers, convince yourself that:
(a) the points (a, b) and $(a, -b)$ are symmetric with respect to the x-axis;
(b) (a, b) and $(-a, b)$ are symmetric with respect to the y-axis;
(c) (a, b) and $(-a, -b)$ are symmetric with respect to the origin.

31 What symmetry statement can be made about the points (a, b) and (b, a)?

32 In each case, place the figure in a convenient position relative to the coordinate system and prove the statement algebraically:
(a) The diagonals of a parallelogram bisect each other.
(b) The sum of the squares of the diagonals of a parallelogram equals the sum of the squares of the sides.
(c) The midpoint of the hypotenuse of a right triangle is equidistant from the three vertices.

Use the fact stated in (c) to show that when the acute angles of a right triangle are 30° and 60°, the side opposite the 30° angle is half the hypotenuse.

33 In an isosceles right triangle, both acute angles are 45°. If the hypotenuse is h, what is the length of each of the other sides?

34 Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be distinct points. If $P = (x, y)$ is on the segment joining P_1 and P_2 and one-third of the way from P_1 to P_2, show that

$$x = \tfrac{1}{3}(2x_1 + x_2) \qquad \text{and} \qquad y = \tfrac{1}{3}(2y_1 + y_2).$$

Find the corresponding formulas if P is two-thirds of the way from P_1 to P_2.

35 Consider an arbitrary triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3). Find the point on each median which is two-thirds of the way from the vertex to the midpoint of the opposite side.* Perform the calculations separately for each median and verify that these three points are all the same, with coordinates

$$\tfrac{1}{3}(x_1 + x_2 + x_3) \qquad \text{and} \qquad \tfrac{1}{3}(y_1 + y_2 + y_3).$$

This proves that the medians of any triangle intersect at a point which is two-thirds of the way from each vertex to the midpoint of the opposite side.

*A *median* of a triangle is a segment joining a vertex to the midpoint of the opposite side.

1.3 SLOPES AND EQUATIONS OF STRAIGHT LINES

In this section we use the language of algebra to describe the set of all points that lie on a given straight line. This algebraic description is called the *equation of the line.* First, however, it is necessary to discuss an important preliminary concept: the slope of a line.

THE SLOPE OF A LINE

Any nonvertical straight line has a number associated with it that specifies its direction, called its *slope*. This number is defined as follows (Fig. 1.9 illustrates the definition). Choose any two distinct points on the line, say $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$. Then the slope is denoted by m and defined to be the ratio

$$m = \frac{y_2 - y_1}{x_2 - x_1}. \tag{1}$$

If we reverse the order of subtraction in both numerator and denominator, then the sign of each is changed, so m is unchanged:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}.$$

This shows that the slope can be computed as the difference of the y-coordinates divided by the difference of the x-coordinates—in either order, as long as both differences are formed in the same order. In Fig. 1.9, where P_2 is placed to the right of P_1 and the line rises to the right, it is clear that the slope as defined by (1) is simply the ratio of the height to the base in the indicated right triangle. It is necessary to know that the value of m depends only on the line itself and is the same no matter where the points P_1 and P_2 happen to be located on the line. This is easy to see by visualizing the effect of moving P_1 and P_2 to different positions on the line; this change gives rise to a similar right triangle and therefore leaves the ratio in (1) unaltered.

If we choose the position of P_2 so that $x_2 - x_1 = 1$, that is, if we place P_2 1 unit to the right of P_1, then $m = y_2 - y_1$. This tells us that the slope is simply the change in y as a point (x, y) moves along the line in such a way that x increases by 1 unit. This change in y can be positive, negative, or zero, depending on the direction of the line. We therefore have the following important correlations between the sign of m and the indicated directions:

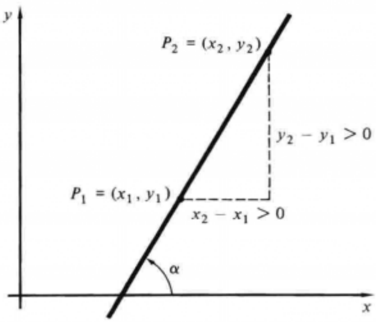

Figure 1.9

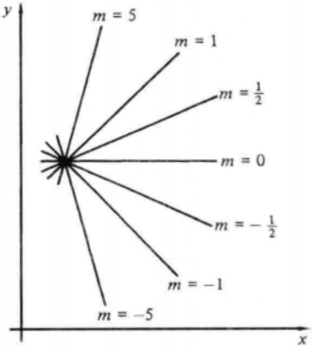

Figure 1.10 A variety of slopes.

$m > 0$, line rises to the right;

$m < 0$, line falls to the right;

$m = 0$, line horizontal.

Further, the absolute value of m is a measure of the steepness of the line (Fig. 1.10). It is evident from (1) why a vertical line has no slope, for in this case the two points have equal x-coordinates and the denominator in (1) is 0—and we know that division by 0 is undefined.

If the line under discussion crosses the x-axis, then the angle α from the positive x-direction to the line, measured counterclockwise, is called the *inclination* —or sometimes the *angle of inclination*—of the line. Students who have studied trigonometry will see from Fig. 1.9 that the slope is the tangent of this angle, $m = \tan \alpha$.

EQUATIONS OF A LINE

A vertical line is characterized by the fact that all points on it have the same x-coordinate. If the line crosses the x-axis at the point $(a, 0)$, then a point (x, y) lies on the line if and only if

$$x = a, \tag{2}$$

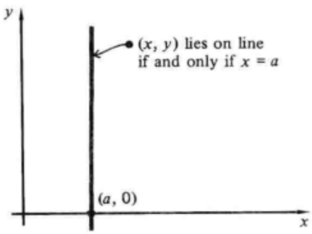

Figure 1.11

as illustrated in Fig. 1.11. The statement that (2) is the equation of the line means precisely this: A point (x, y) lies on the line if and only if condition (2) is satisfied.

Next consider a nonvertical line, and let it be "given" in the sense that we know a point (x_0, y_0) on it and its slope m (Fig. 1.12). If (x, y) is a point in the plane that does not lie on the vertical line through (x_0, y_0), then it is easy to see that this point lies on the given line if and only if the line determined by (x_0, y_0) and (x, y) has the same slope as the given line:

$$\frac{y - y_0}{x - x_0} = m. \tag{3}$$

This would be the equation of our line except for the minor flaw that the coordinates of the point (x_0, y_0)—which is certainly on the line—do not satisfy the equation (they reduce the left side to the meaningless expression 0/0). This flaw is easily removed by writing equation (3) in the form

$$y - y_0 = m(x - x_0). \tag{4}$$

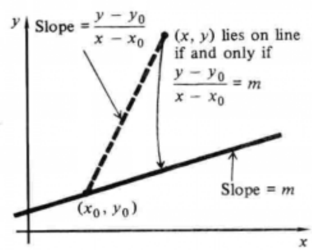

Figure 1.12

Nevertheless, we usually prefer the form (3), because its direct connection with the geometric idea illustrated in Fig. 1.12 makes it easy to remember. Either equation (or both) is called the *point-slope equation* of a line, since the line is initially specified by means of a known point on it and its known slope. To grasp more firmly the meaning of equation (4), imagine a point (x, y) moving along the given line. As this point moves, its coordinates x and y change; but even though they change, they are bound together by the fixed relationship expressed by equation (4).

If the known point on the line happens to be the point where the line crosses the y-axis, and if this point is denoted by $(0, b)$, then equation (4) becomes $y - b = mx$ or

$$y = mx + b. \tag{5}$$

The number b is called the *y-intercept* of the line, and (5) is called the *slope-intercept equation* of a line. This form is especially convenient because it tells at a glance the location and direction of a line. For example, if the equation

$$6x - 2y - 4 = 0 \tag{6}$$

is solved for y, we see that

$$y = 3x - 2. \tag{7}$$

Comparing (7) with (5) shows at once that $m = 3$ and $b = -2$, and so (6) and (7) both represent the line that passes through $(0, -2)$ with slope 3. This information makes it very easy to sketch the line. It may seem that (6) and (7) are different equations, so that (6) should be referred to as "an" equation of the line and (7) as "another" equation of the line, but we prefer to regard them as merely different forms of a single equation. Many other forms are possible, for instance,

$$y + 2 = 3x, \qquad x = \tfrac{1}{3}y + \tfrac{2}{3}, \qquad 3x - y = 2.$$

It is reasonable to cut through appearances and speak of any one of these as "the" equation of the line.

More generally, every equation of the form

$$Ax + By + C = 0, \tag{8}$$

where the constants A and B are not both zero, represents a straight line. For if $B = 0$, then $A \neq 0$, and the equation can be written as

$$x = -\frac{C}{A},$$

which is clearly the equation of a vertical line. On the other hand, if $B \neq 0$, then

$$y = -\frac{A}{B}x - \frac{C}{B},$$

and this equation has the form (5) with $m = -A/B$ and $b = -C/B$. Equation (8) is rather inconvenient for most purposes because its constants are not directly related to the geometry of the line. Its main merit is that it is capable of representing all lines, without any need for distinguishing between the vertical and nonvertical cases. For this reason it is called the *general linear equation.*

PARALLEL AND PERPENDICULAR LINES

Two distinct nonvertical straight lines with slopes m_1 and m_2 are evidently parallel if and only if their slopes are equal:

$$m_1 = m_2.$$

The criterion for perpendicularity is the relation

$$m_1 m_2 = -1. \tag{9}$$

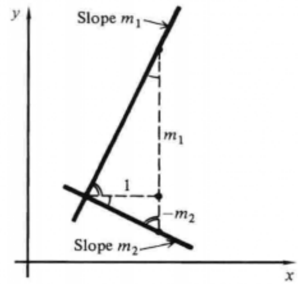

Figure 1.13

This is not obvious, but can be established quite easily by using similar triangles, as follows (Fig. 1.13). Suppose that the lines are perpendicular, as shown in Fig. 1.13. Draw a segment of length 1 to the right from their point of intersection, and from its right endpoint draw vertical segments up and down to the two lines. From the meaning of the slopes, the two right triangles formed in this way have sides of the indicated lengths. Since the lines are perpendicular, the indicated angles are equal and the triangles are similar. This similarity implies that the following ratios of corresponding sides are equal:

$$\frac{m_1}{1} = \frac{1}{-m_2}.$$

This is equivalent to (9), so (9) is true when the lines are perpendicular. The reasoning given here is easily reversed, telling us that if (9) is true, then the lines are perpendicular. Since equation (9) is equivalent to

$$m_1 = -\frac{1}{m_2} \quad \text{and} \quad m_2 = -\frac{1}{m_1},$$

we see that two nonvertical lines are perpendicular if and only if their slopes are negative reciprocals of one another.

The ideas of this section enlarge our supply of tools for proving geometric theorems by algebraic methods.

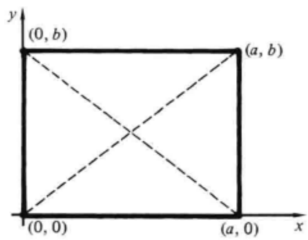

Figure 1.14

Example If the diagonals of a rectangle are perpendicular, then the rectangle is a square. To establish this, we place the rectangle in the convenient position shown in Fig. 1.14. The slopes of the diagonals are clearly b/a and $-b/a$. If these diagonals are perpendicular, then

$$\frac{b}{a} = \frac{a}{b}, \qquad a^2 = b^2, \qquad a^2 - b^2 = 0, \qquad \text{and} \qquad (a+b)(a-b) = 0.$$

The last equation implies that $a = b$, so the rectangle is a square.

PROBLEMS

1 Plot each pair of points, draw the line they determine, and compute the slope of this line:
(a) $(-3, 1), (4, -1)$; (b) $(2, 7), (-1, -1)$;
(c) $(-4, 0), (2, 1)$; (d) $(-4, 3), (5, -6)$;
(e) $(-5, 2), (7, 2)$; (f) $(0, -4), (1, 6)$.

2 Plot each of the following sets of three points, and use slopes to determine in each case whether all three points lie on a single straight line:
(a) $(5, -1), (2, 2), (-4, 6)$;
(b) $(1, 1), (-5, -2), (5, 3)$;
(c) $(4, 3), (10, 14), (-2, -8)$;
(d) $(-1, 3), (6, -1), (-9, 7)$.

3 Plot the points $(-1, -1)$, $(9, 1)$, $(8, 6)$, and $(-2, 4)$, and show that they are the vertices of a rectangle.

4 Plot the points $(-3, 8)$, $(3, 5)$, $(0, -1)$, and $(-6, 2)$, and show that they are the vertices of a square.

5 Plot each of the following sets of three points, and use slopes to determine in each case whether the points form a right triangle:
(a) $(2, -3), (5, 2), (0, 5)$;
(b) $(10, -5), (5, 4), (-7, -2)$;
(c) $(8, 2), (-1, -1), (2, -7)$;
(d) $(-2, 6), (3, -4), (8, 11)$.

6 Write the equation of each line in Problem 1 using the point-slope form; then rewrite each of these equations in the form $y = mx + b$ and find the y-intercept.

7 Find the equation of the line:
(a) through $(2, -3)$ with slope -4;
(b) through $(-4, 2)$ and $(3, -1)$;
(c) with slope $\frac{2}{3}$ and y-intercept -4;
(d) through $(2, -4)$ and parallel to the x-axis;
(e) through $(1, 6)$ and parallel to the y-axis;

(f) through $(4, -2)$ and parallel to $x + 3y = 7$;
(g) through $(5, 3)$ and perpendicular to $y + 7 = 2x$;
(h) through $(-4, 3)$ and parallel to the line determined by $(-2, -2)$ and $(1, 0)$;
(i) that is the perpendicular bisector of the segment joining $(1, -1)$ and $(5, 7)$;
(j) through $(-2, 3)$ with inclination $135°$.

8 If a line crosses the x-axis at the point $(a, 0)$, the number a is called the *x-intercept* of the line. If a line has x-intercept $a \neq 0$ and y-intercept $b \neq 0$, show that its equation can be written as

$$\frac{x}{a} + \frac{y}{b} = 1.$$

This is called the *intercept form* of the equation of a line. Notice that it is easy to put $y = 0$ and see that the line crosses the x-axis at $x = a$, and to put $x = 0$ and see that the line crosses the y-axis at $y = b$.

9 Put each equation in intercept form and sketch the corresponding line:
(a) $5x + 3y + 15 = 0$; (b) $3x = 8y - 24$;
(c) $y = 6 - 6x$; (d) $2x - 3y = 9$.

10 The set of all points (x, y) that are equally distant from the points $P_1 = (-1, -3)$ and $P_2 = (5, -1)$ is the perpendicular bisector of the segment joining these points. Find its equation
(a) by equating the distances from (x, y) to P_1 and P_2, and simplifying the resulting equation;
(b) by finding the midpoint of the given segment and using a suitable slope.

11 Sketch the lines $3x + 4y = 7$ and $x - 2y = 6$, and find their point of intersection. Hint: Their point of intersection is that point (x, y) whose coordinates satisfy both equations simultaneously.

12 Find the point of intersection of each of the following pairs of lines:
(a) $2x + 2y = 2$, $y = x - 1$;
(b) $10x + 7y = 24$, $15x - 4y = 7$;
(c) $3x - 5y = 7$, $15y + 25 = 9x$.

13 Let F and C denote temperature in degrees Fahrenheit and degrees Celsius. Find the equation connecting F and C, given that it is linear and that $F = 32$ when $C = 0$, $F = 212$ when $C = 100$.

14 Find the values of the constant k for which the line $(k - 3)x - (4 - k^2)y + k^2 - 7k + 6 = 0$ is
(a) parallel to the x-axis;
(b) parallel to the y-axis;
(c) through the origin.

15 Show that the segments joining the midpoints of adjacent sides of any quadrilateral form a parallelogram.

16 Show that the lines from any vertex of a parallelogram to the midpoints of the opposite sides trisect a diagonal.

17 Let $(0, 0)$, $(a, 0)$, and (b, c) be the vertices of an arbitrary triangle placed so that one side lies along the positive x-axis with its left endpoint at the origin. If the square of this side equals the sum of the squares of the other two sides, use slopes to show that the triangle is a right triangle. Thus, the converse of the Pythagorean theorem is also true.

1.4 CIRCLES AND PARABOLAS. DESCARTES AND FERMAT

The coordinate plane or xy-plane is often called the *Cartesian plane,* and x and y are frequently referred to as the *Cartesian coordinates* of the point $P = (x, y)$. The word "Cartesian" comes from Cartesius, the Latinized name of the French philosopher-mathematician Descartes, who is considered one of the two principal founders of analytic geometry.* The basic idea of this subject is quite simple: Exploit the correspondence between points and their coordinates to study geometric problems—especially the properties of curves—with the tools of algebra. The reader will see this idea in action throughout this book. Generally speaking, geometry is visual and intuitive, while algebra is rich in computational machinery, and each can serve the other in many fruitful ways.

Most people who have had a course in algebra have learned that an equation

$$F(x, y) = 0 \tag{1}$$

usually determines a curve (its *graph*) which consists of all points $P = (x, y)$ whose coordinates satisfy the given equation. Conversely, a curve defined by some geometric condition can usually be described algebraically by an equation

*The other (also French) was Fermat, a less well known figure than Descartes but a much greater mathematician. The names of these two men are pronounced "Fair-MA" and "Day-CART."

of the form (1). It is intuitively clear that straight lines are the simplest curves, and our work in Section 1.3 demonstrated that straight lines in the coordinate plane correspond to linear equations in x and y. We now develop algebraic descriptions of several other curves that will be useful as illustrative examples in the next few chapters.

CIRCLES

The distance formula of Section 1.2 is often useful in finding the equation of a curve whose geometric definition depends on one or more distances.

One of the simplest curves of this kind is a *circle,* which can be defined as the set of all points at a given distance (the radius) from a given point (the center). If the center is the point (h, k) and the radius is the positive number r (Fig. 1.15), and if (x, y) is an arbitrary point on the circle, then the defining condition says that

$$\sqrt{(x-h)^2 + (y-k)^2} = r.$$

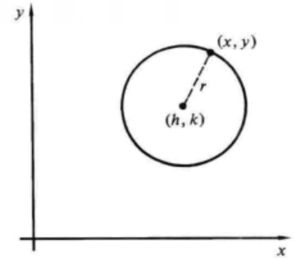

Figure 1.15 Circle.

It is convenient to eliminate the radical sign by squaring, which yields

$$(x-h)^2 + (y-k)^2 = r^2. \tag{2}$$

This is therefore the equation of the circle with center (h, k) and radius r. In particular, if the center happens to be the origin, so that $h = k = 0$, then

$$x^2 + y^2 = r^2$$

is the equation of the circle.

Example 1 If the radius of a circle is $\sqrt{10}$ and its center is $(-3, 4)$, then its equation is

$$(x+3)^2 + (y-4)^2 = 10.$$

Notice that the coordinates of the center are the numbers *subtracted* from x and y in the parentheses.

Example 2 An angle inscribed in a semicircle is necessarily a right angle.* To prove this algebraically, let the semicircle have radius r and center at the origin (Fig. 1.16), so that its equation is $x^2 + y^2 = r^2$ with $y \geq 0$. The inscribed angle is a right angle if and only if the product of the slopes of its sides is -1, that is,

$$\frac{y}{x-r} \cdot \frac{y}{x+r} = -1. \tag{3}$$

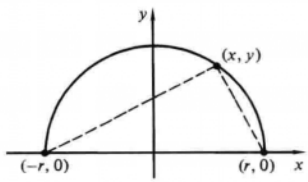

Figure 1.16

This is easily seen to be equivalent to $x^2 + y^2 = r^2$, which is certainly true for any point (x, y) on the semicircle, so (3) is true and the angle is a right angle.

It is clear that any equation of the form (2) is easy to interpret geometrically. For instance,

*According to tradition, this is one of the theorems discovered and proved by Thales.

$$(x-5)^2 + (y+2)^2 = 16 \tag{4}$$

is immediately recognizable as the equation of the circle with center $(5, -2)$ and radius 4, and this information enables us to sketch the graph without difficulty. However, if the equation has been roughly treated by someone who likes to "simplify" things algebraically, then it might have the form

$$x^2 + y^2 - 10x + 4y + 13 = 0. \tag{5}$$

This is an equivalent but scrambled version of (4), and its constants tell us nothing directly about the nature of the graph. To find out what the graph is, we must "unscramble" by *completing the square.** To do this, we begin by rewriting equation (5) as

$$(x^2 - 10x + \quad) + (y^2 + 4y + \quad) = -13,$$

with the constant term moved to the right and blank spaces provided for the insertion of suitable constants. When the square of half the coefficient of x is added in the first blank space and the square of half the coefficient of y in the second, and the same constants are added to the right side to maintain the balance of the equation, we get

$$(x^2 - 10x + 25) + (y^2 + 4y + 4) = -13 + 25 + 4$$

or

$$(x-5)^2 + (y+2)^2 = 16. \tag{6}$$

Exactly the same process can be applied to the general equation of the form (5), namely,

$$x^2 + y^2 + Ax + By + C = 0, \tag{7}$$

but there is little to be gained by writing out the details in this general case. However, it is important to notice that if the constant term 13 in (5) is replaced by 29, then (6) becomes

$$(x-5)^2 + (y+2)^2 = 0,$$

whose graph is the single point $(5, -2)$. Similarly, if this constant term is replaced by any number greater than 29, then the right-hand side of (6) becomes negative and the graph is empty, in the sense that there are no points (x, y) in the plane whose coordinates satisfy the equation. We therefore see that the graph of (7) is sometimes a circle, sometimes a single point, and sometimes empty—depending entirely on the constants A, B, and C.

PARABOLAS

The definition we use for a *parabola* is the following (Fig. 1.17*a*): It is the curve consisting of all points that are equally distant from a fixed point F (called the *focus*) and a fixed line d (called the *directrix*). The distance from a point to a line is always understood to mean the perpendicular distance.

*The form of the equation $(x+a)^2 = x^2 + 2ax + a^2$ is the key to the process of completing the square. Notice that the right side is a perfect square—the square of $x + a$—precisely because its constant term is the square of half the coefficient of x.

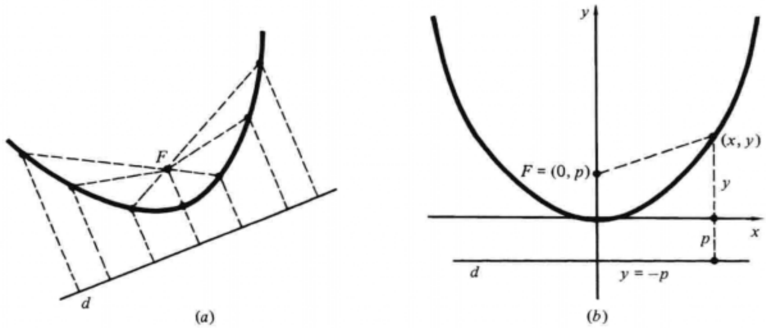

Figure 1.17 Parabola.

To find a simple equation for a parabola, we place it in the coordinate system as shown in Fig. 1.17*b*, with the focus and directrix equally far above and below the x-axis. The line through the focus perpendicular to the directrix is called the *axis* of the parabola; this is the axis of symmetry of the curve, and is the y-axis in the figure. The point on the axis halfway between the focus and the directrix is called the *vertex* of the parabola; in the figure this point is the origin. If (x, y) is an arbitrary point on the parabola, the condition expressed in the definition is stated algebraically by the equation

$$\sqrt{x^2 + (y - p)^2} = y + p. \tag{8}$$

On squaring both sides and simplifying, we obtain

$$x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2$$

or

$$x^2 = 4py. \tag{9}$$

These steps are reversible, so (8) and (9) are equivalent and (9) is the equation of the parabola whose focus and directrix are located as shown in Fig. 1.17*b*. Notice particularly that the positive constant p in (9) is the distance from the focus to the vertex, and also from the vertex to the directrix.

If we change the position of the parabola relative to the coordinate axes, we naturally change its equation. Three other positions are shown in Fig. 1.18, each with its corresponding equation and with $p > 0$ in each case. Students should

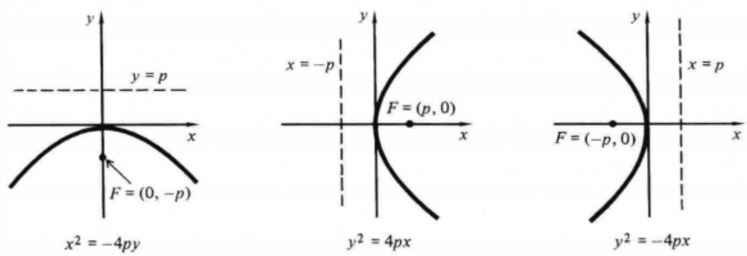

Figure 1.18 Various parabolas.

verify the correctness of all three equations. We also point out that each of these four equations can be put in the form

$$y = ax^2 \tag{10}$$

or

$$x = ay^2.$$

These forms conceal the constant p, with its geometric significance, but as compensation they are more useful in visualizing the overall appearance of the graph. For instance, in (10) the variable x is squared but y is not. This tells us that as a point (x, y) moves out along the curve, y increases much faster than x, and so the curve opens in the y-direction—upward or downward, according as a is positive or negative. It also tells us that the graph is symmetric with respect to the y-axis, because x is squared, and therefore we get the same number y for any number x and its negative.

Example 3 What is the graph of the equation $12x + y^2 = 0$? If this is put in the form $y^2 = -12x$ and compared with the equation on the right in Fig. 1.18, it is clear that the graph is a parabola with vertex at the origin and opening to the left. Since $4p = 12$ and therefore $p = 3$, the point $(-3, 0)$ is the focus and $x = 3$ is the directrix.

Example 4 The graph of $y = 2x^2$ is evidently a parabola with vertex at the origin and opening upward. To find its focus and directrix, the equation must be rewritten as $x^2 = \frac{1}{2}y$ and compared with equation (9). This yields $4p = \frac{1}{2}$, so $p = \frac{1}{8}$. The focus is therefore $(0, \frac{1}{8})$, and the directrix is $y = -\frac{1}{8}$.

We illustrate one last point about parabolas by examining the equation

$$y = x^2 - 4x + 5. \tag{11}$$

If this is written as

$$y - 5 = x^2 - 4x,$$

and if we complete the square on the terms involving x, then the result is

$$y - 1 = (x - 2)^2. \tag{12}$$

If we now introduce the new variables

$$\begin{aligned} X &= x - 2, \\ Y &= y - 1, \end{aligned} \tag{13}$$

then equation (12) becomes

$$Y = X^2.$$

The graph of this equation is clearly a parabola opening upward with vertex at the origin of the XY coordinate system. By equations (13), the origin in the XY system is the point $(2, 1)$ in the xy system, as shown in Fig. 1.19. What has happened here is that the coordinate system has been shifted or translated to a new position in the plane, and the axes renamed, and equations (13) express the re-

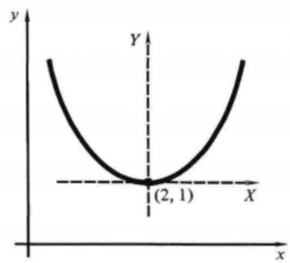

Figure 1.19

lation between the coordinates of an arbitrary point with respect to each of the two coordinate systems. In exactly the same way, any equation of the form

$$y = ax^2 + bx + c, \qquad a \neq 0,$$

represents a parabola with vertical axis which is congruent to $y = ax^2$ and opens up or down according as the number a is positive or negative. Similarly, the equation

$$x = ay^2 + by + c, \qquad a \neq 0,$$

represents a parabola with horizontal axis which opens to the right or left according as $a > 0$ or $a < 0$.

In our work up to this stage we have used the static concept of a curve as a certain set of points or geometric figure. It is often possible to adopt the dynamic point of view, in which a curve is thought of as the path of a moving point. For instance, a circle is the path of a point that moves in such a way that it maintains a fixed distance from a given point. When this mode of thought is used—with its advantage of greater intuitive vividness—a curve is often called a *locus.* Thus, a parabola is the locus of a point that moves in such a way that it maintains equal distances from a given point and a given line.

NOTE ON DESCARTES AND FERMAT

Are there some people among us who feel that what passes for "knowledge" in our time is an uncritical mishmash of sense and nonsense, fact and guesswork, gossip and hearsay and clumsy propaganda—mostly acquired from wishful thinking, lazy reasoning, inadequate senses, credulous parents, overworked teachers, and self-serving institutions? This was also the opinion of the 23-year-old Frenchman René Descartes (1596–1650) on Nov. 10, 1619. For this was the day above all others when the modern world began, our world of victorious rationality and triumphant science.

On this day—a famous day in the history of thought—in a state of exhaustion and feverish excitement, Descartes found the method he sought for extending the certainty of mathematics to all other fields of knowledge:

> The long chains of simple reasoning which geometers use to arrive at their most difficult conclusions made me believe that all things which are the objects of human knowledge are similarly interdependent; and that if we will only abstain from assuming something to be true which is not, and always follow the necessary order in deducing one thing from another, there is nothing so remote that we cannot reach it, nor so hidden that we cannot discover it.

This is a quotation from Part 2 of his *Discourse on Method,* a short and highly readable book published in 1637 which is commonly considered to mark the birth of modern philosophy. In this work he rejected the sterile scholasticism prevailing at the time and set himself the task of rebuilding knowledge from the ground up, on a foundation of reason and science instead of authority and faith. He provided the fresh points of view needed for the vigorous development of the Scientific Revolution, whose influence has been the dominant fact of modern history. Further, in an appendix to the *Discourse* on his ideas about geometry, he foreshadowed the new forms of mathematics—analytic geometry and calculus—without which this Revolution would have died in infancy. It was no exaggeration for the great American jurist Oliver Wendell Holmes to write: "Descartes commanded the future from his study more than Napoleon from his throne."

Descartes was a brilliant man—and enormously influential with a corresponding ego—but he was not quite as brilliant as he thought. His contemporary Pierre Fermat (1601–1665) was a man of genius and perhaps the greatest mathematician of the seventeenth century; and when the two men collided on issues of science or mathematics, it was always Descartes's nose that was bloodied.

By profession Fermat was a lawyer and a member of the provincial supreme court in Toulouse, a city in southwestern France. However, his hobby and private passion was mathematics, and his casual creativity was one of the wonders of the age to the few who knew about it. His letters suggest that he was a shy and retiring man, courteous and affable but slightly remote. His outward life was as quiet and orderly as one would expect of a provincial judge with a sense of responsibility toward his work. Fortunately this work was not too demanding, and left ample leisure for the extraordinary inner life that flourished by lamplight in the silence of his study at night.

He invented analytic geometry in 1629 and described his ideas in a short work that circulated in manuscript from early 1637 on, but was not published in his lifetime. The credit for this achievement has usually been given to Descartes on the basis of his *Geometry*, which was published late in 1637 as an appendix to his *Discourse on Method*. However, nothing that we would recognize as analytic geometry can be found in Descartes's essay, except perhaps the idea of using algebra as a language for discussing geometric problems. Fermat had the same idea but did something important with it: He introduced perpendicular axes and found the general equations of straight lines and circles and the simplest equations of parabolas, ellipses, and hyperbolas; and he further showed in a fairly complete and systematic way that every first- or second-degree equation can be reduced to one of these types. Descartes certainly knew some analytic geometry by the late 1630s; but since he had possession of the original manuscript of Fermat's short essay (of which Fermat himself did not bother to keep a copy) several months before the publication of his own *Geometry*, it is likely that much of what he knew he learned from Fermat.

The invention of calculus is usually credited to Newton and Leibniz, whose ideas and methods were not published until about 20 years after Fermat's death. However, if differential calculus is considered to be the mathematics of finding maxima and minima of functions and drawing tangents to curves, then Fermat was the true creator of this subject as early as 1629, more than a decade before either Newton or Leibniz was born. With his usual honesty in such matters, Newton stated—in a letter that was discovered only in 1934—that his own early ideas about calculus came directly from "Fermat's way of drawing tangents."

Fermat was also the founder of mathematical optics and the joint founder (in correspondence with Blaise Pascal) of the theory of probability. But to him all these activities were of minor importance compared with the consuming passion of his life, the theory of numbers. It was here that his genius shone most brilliantly, for his insight into the properties of the familiar but mysterious positive integers has perhaps never been equaled. He was the sole and undisputed founder of the modern era in this important branch of pure mathematics, without any rivals and with few followers until the next century.

To illustrate the nature of his achievement in number theory, we mention his profound and beautiful *four squares theorem:* Every positive integer is either a square or the sum of two, three, or four squares. Like many of his discoveries, this was jotted down in the margin of one of his books, and his proof went unrecorded and was lost forever when he died. A proof was found at last in 1772—more than a century after Fermat's death—as the culmination of 40 years of effort by one of the greatest mathematicians of the eighteenth century. As we see, mathematicians are people who are not only irresistibly attracted by truths of this kind but also cannot rest until they know *why* they are true.

Without visibly trying, and as naturally as a hawk sustains itself on the wind, Fermat attained immortal fame among mathematicians. There are many reasons for this immortality, one of the most interesting being the legacy of what is now known as *Fermat's last theorem:* If $n > 2$, then the equation $x^n + y^n = z^n$ has no positive integer solutions x, y, z. Again, he wrote this statement in the margin of a book he was studying, near a passage dealing with the fact that $x^2 + y^2 = z^2$ has many solutions—3, 4, 5 and 5, 12, 13, among others. He then added the tantalizing remark, "I have found a truly wonderful proof which this margin is too narrow to contain." Unfortunately no proof has ever been discovered by anyone else, and Fermat's last theorem remains to this day one of the most baffling unsolved problems of mathematics.*

*Late report from the cutting edge: It appears that Fermat's last theorem may have been proved by Andrew Wiles of Princeton University. This was announced on June 23, 1993, in the last of three lectures Wiles gave at Cambridge University, in England. The proof is about 200 pages long and follows a tortuous, roundabout path through many tangled jungles of sophisticated pure mathematics. The careful checking of every line of this proof may take years to carry out. It is estimated that perhaps a tenth of 1 percent of mathematicians could understand all details of the proof—and this definitely does not include the present writer. If Wiles's proof checks out, the challenge will still remain of discovering a one- or two-page (or even a three- or four-page) proof of Fermat's one-sentence theorem. For further details, see *Newsweek*, July 5, 1993, or *Scientific American*, September 1993.

PROBLEMS

1 Find the equation of the circle with the given point as center and the given number as radius:
(a) $(4, 6)$, 3; (b) $(-3, 7)$, $\sqrt{5}$;
(c) $(-5, -9)$, 7; (d) $(1, -6)$, $\sqrt{2}$;
(e) $(a, 0)$, a; (f) $(0, a)$, a.

2 In each case find the equation of the circle determined by the given conditions:
(a) Center $(2, 3)$ and passes through $(-1, -2)$.
(b) The ends of a diameter are $(-3, 2)$ and $(5, -8)$.
(c) Center $(4, 5)$ and tangent to the x-axis.
(d) Center $(-4, 1)$ and tangent to the line $x = 3$.
(e) Center $(-2, 3)$ and tangent to the line $4y - 3x + 2 = 0$.
(f) Center on the line $x + y = 1$, passes through $(-2, 1)$ and $(-4, 3)$.
(g) Center on the line $y = 3x$ and tangent to the line $x = 2y$ at the point $(2, 1)$.

3 In each of the following, determine the nature of the graph of the given equation by completing the square:
(a) $x^2 + y^2 - 4x - 4y = 0$.
(b) $x^2 + y^2 - 18x - 14y + 130 = 0$.
(c) $x^2 + y^2 + 8x + 10y + 40 = 0$.
(d) $4x^2 + 4y^2 + 12x - 32y + 37 = 0$.
(e) $x^2 + y^2 - 8x + 12y + 53 = 0$.
(f) $x^2 + y^2 - \sqrt{2}x + \sqrt{2}y + 1 = 0$.
(g) $x^2 + y^2 - 16x + 6y - 48 = 0$.

4 Find the equation of the locus of a point $P = (x, y)$ that moves in accordance with each of the following conditions, and sketch the graphs:
(a) The sum of the squares of the distances from P to the points $(a, 0)$ and $(-a, 0)$ is $4b^2$, where $b \geq a/\sqrt{2} > 0$.
(b) The distance of P from the point $(8, 0)$ is twice its distance from the point $(0, 4)$.

5 The *quadratic formula* for the roots of the quadratic equation $ax^2 + bx + c = 0$ is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Derive this formula from the equation by dividing through by a, moving the constant term to the right side, and completing the square. Under what circumstances does the equation have distinct real roots, equal real roots, and no real roots?

6 At what points does the circle $x^2 + y^2 - 8x - 6y - 11 = 0$ intersect
(a) the x-axis? (b) the y-axis?
(c) the line $x + y = 1$?
Sketch the figure, and use this picture to judge whether your answers are reasonable or not.

7 Find the equations of all lines that are tangent to the circle $x^2 + y^2 = 2y$ and pass through the point $(0, 4)$. Hint: The line $y = mx + 4$ is tangent to the circle if it intersects the circle at only one point.

8 Find the focus and directrix of each of the following parabolas, and sketch the curves:
(a) $y^2 = 12x$; (b) $y = 4x^2$;
(c) $2x^2 + 5y = 0$; (d) $4x + 9y^2 = 0$;
(e) $x = -2y^2$; (f) $12y = -x^2$;
(g) $16y^2 = x$; (h) $24x^2 = y$;
(i) $y^2 + 8y - 16x = 16$; (j) $x^2 + 2x + 29 = 7y$.

9 Sketch the parabola and find its equation if it has
(a) vertex $(0, 0)$ and focus $(-3, 0)$;
(b) vertex $(0, 0)$ and directrix $y = -1$;
(c) vertex $(0, 0)$ and directrix $x = -2$;
(d) vertex $(0, 0)$ and focus $(0, -\frac{1}{3})$;
(e) directrix $x = 2$ and focus $(-4, 0)$;
(f) focus $(3, 3)$ and directrix $y = -1$.

10 Find the focus and directrix of each of the following parabolas, and sketch the curves:
(a) $y = x^2 + 1$; (b) $y = (x - 1)^2$;
(c) $y = (x - 1)^2 + 1$; (d) $y = x^2 - x$.

11 Water squirting out of a horizontal nozzle held 4 ft above the ground describes a parabolic curve with the vertex at the nozzle. If the stream of water drops 1 ft in the first 10 ft of horizontal motion, at what horizontal distance from the nozzle will it strike the ground?

12 Show that there is exactly one line with given slope m which is tangent to the parabola $x^2 = 4py$, and find its equation.

13 Prove that the two tangents to a parabola from any point on the directrix are perpendicular.

1.5 THE CONCEPT OF A FUNCTION

The most important concept in all of mathematics is that of a function. No matter what branch of the subject we consider—algebra, geometry, number theory, probability, or any other—it almost always turns out that functions are the primary objects of investigation. This is particularly true of calculus, in which most of our work will be concerned with constructing machinery for the study of functions and applying this machinery to problems in science and geometry.

What is a function? Briefly—and we expand on this below—if x and y are two variables that are related in such a way that whenever a permissible numerical value is assigned to x, there is determined one and only one corresponding numerical value for y, then y is called *a function of* x.

Example 1 (a) If a rock is dropped from the edge of a cliff, and it falls s feet in t seconds, then s is a function of t. It is known from experiment that (approximately) $s = 16t^2$.

(b) The area A of a circle is a function of its radius r. It is known from geometry that $A = \pi r^2$.

(c) If the manager of a bookstore buys n books from a publisher at \$12 per copy and the shipping charges are \$35, then his cost C for these books is a function of n given by the formula $C = 12n + 35$.

We continue building our understanding of the concept of a function by considering an example directly related to our work in the preceding section.

Example 2 We examine the equation

$$y = x^2$$

and its corresponding graph, which we know is a parabola that opens upward and has its vertex at the origin (Fig. 1.20). In Section 1.4 we thought of this equation as a relation between the variable coordinates of a point (x, y) moving along the curve. We now shift our point of view, and instead think of it as a formula that provides a mechanism for calculating the numerical value of y when the numerical value of x is given. Thus, $y = 1$ when $x = 1$, $y = 4$ when $x = 2$, $y = \frac{1}{4}$ when $x = \frac{1}{2}$, $y = 1$ when $x = -1$, and so on. The value of y is therefore said to *depend on,* or to be *a function of,* the value of x. This dependence can be expressed in functional notation by writing

$$y = f(x) \qquad \text{where} \qquad f(x) = x^2.$$

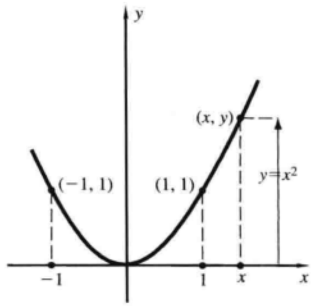

Figure 1.20

The symbol $f(x)$ is read "f of x," and the letter f represents the rule or process–squaring, in this particular case—which is applied to any number x to yield the corresponding number y. The numerical examples just given can therefore be written as $f(1) = 1$, $f(2) = 4$, $f(\frac{1}{2}) = \frac{1}{4}$, and $f(-1) = 1$. The meaning of this notation can perhaps be further clarified by observing that

$$f(x+1) = (x+1)^2 = x^2 + 2x + 1 \qquad \text{and} \qquad f(x^3) = (x^3)^2 = x^6;$$

that is, the rule f simply produces the square of whatever quantity follows it in parentheses.

This example suggests the general concept of a function as we shall use it in most of our work. We formulate this concept as follows.

Let D be a given set of real numbers. A *function* f defined on D is a formula, or rule, or law of correspondence that assigns a single real number y to each number x in D. The set D of allowed values of x is called the *domain* (or *domain of definition*) of the function, and the set of corresponding values of y is called its *range.* The number y that is assigned to x by the function f is usually written $f(x)$ —so that $y = f(x)$—and is called the *value* of f at x. It is customary to call x the *independent variable* because it is free to assume any value in the domain, and to call y the *dependent variable* because its numerical value depends on the choice of x.

There is nothing illegal or immoral about using other letters than x and y to denote the variables. In Example 1, for instance, the independent variables are t, r, and n, and the dependent variables are s, A, and C. Also, as we see in the next example, there is nothing sacred about the letter f, and other letters can be used to designate functions.

Example 3 (a) If a function $f(x)$ is defined by the formula $f(x) = x^3 - 3x^2 + 5$, then $f(2) = 2^3 - 3 \cdot 2^2 + 5 = 1$, $f(0) = 5$, and $f(-2) = (-2)^3 - 3(-2)^2 + 5 = -15$.

(b) If a function $g(x)$ is defined by the formula $g(x) = \sqrt{x}$, then $g(1) = \sqrt{1} = 1$, $g(4) = \sqrt{4} = 2$, and a calculator tells us that $g(10) = \sqrt{10} = 3.16227766017$, approximately. In this case the only allowed values of x are those for which $x \geq 0$, because square roots of negative numbers are not real numbers.

(c) If a function $h(x)$ is defined by the formula $h(x) = 1/(4 - x)$, then $h(1) = 1/(4 - 1) = \frac{1}{3}$, $h(2) = 1/(4 - 2) = \frac{1}{2}$, and $h(4) = 1/(4 - 4) = \frac{1}{0}$ does not exist, because division by zero is not permitted in algebra. Thus, $x = 4$ is the only value of x that is not allowed.

We point out that a function is not fully known until we know precisely which real numbers are permissible values for the independent variable x. The domain is therefore an indispensable part of the concept of a function. In practice, however, most of the specific functions we deal with are defined only by formulas like the ones in Example 3, and nothing is said about the domain. Unless we state otherwise, the domain of such a function is understood to be the set of all real numbers x for which the formula makes sense. In part (a) of Example 3, this means all real numbers; in (b), all real numbers $x \geq 0$; and in (c), all real numbers except $x = 4$.

The reader is undoubtedly acquainted with the idea of the *graph* of a function f: If we imagine the domain D spread out on the x-axis in the coordinate plane (Fig. 1.21*a*), then to each number x in D there corresponds a number $y = f(x)$, and the set of all the resulting points (x, y) in the plane is the graph. Graphs are pictures of functions that enable us to see these functions in their entirety, and we will examine many in the next section.

Many people find it helpful to visualize a function by means of a *machine diagram,* as shown in Fig. 1.21*b*. Here a number x in the domain is fed into the machine, where it is acted upon by the specific instructions built into the function f, and this action produces the resulting number $f(x)$. The domain is the set of all permissible inputs x, and the range is the set of all outputs $f(x)$.

Another way to picture a function is by an *arrow diagram,* in which the domain is thought of as a certain set of points on the page and the range as another set of points (Fig. 1.21*c*). The arrow shows that x has $f(x)$ corresponding to it, and the function f is the complete collection of all these correspondences thought of as a mapping of the first set onto the second.

We mention machine diagrams and arrow diagrams *only* to help students who may be having difficulty grasping the concept of a function. The basic tool for visualizing functions throughout our work will always be graphs. Also, we will

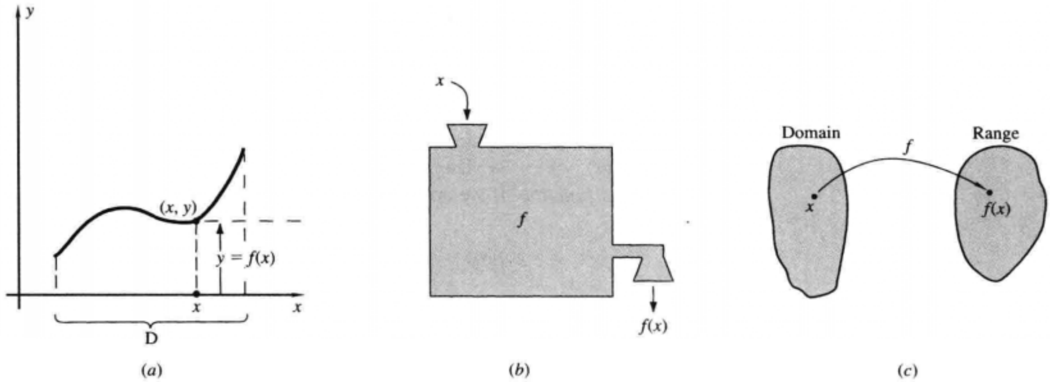

Figure 1.21

see in Section 2.1 that graphs are essential for formulating the main purposes of calculus.

Originally, the only functions mathematicians considered were those defined by formulas. This led to the useful intuitive idea that a function f "does something" to each number x in its domain to "produce" the corresponding number $y = f(x)$. Thus, if

$$y = f(x) = (x^3 + 4)^2,$$

then y is the result of applying certain specific operations to x: Cube it, add 4, and square the sum. On the other hand, the following is also a perfectly legitimate function which is defined by a verbal prescription instead of a formula:

$$y = f(x) = \begin{cases} 1 & \text{if } x \text{ is a rational number,} \\ 0 & \text{if } x \text{ is an irrational number.} \end{cases}$$

All that is really required of a function is that y be uniquely determined—in any manner whatever—when x is specified; beyond this, nothing is said about the nature of the rule f. In discussions that focus on ideas instead of specific functions, such broad generality is often an advantage. We will understand this better in Chapter 6, where one of our problems is to discover what conditions must be imposed on an arbitrary function to guarantee that its integral exists.

An additional remark on usage is perhaps in order. Strictly speaking, the word "function" refers to the rule of correspondence f that assigns a unique number $y = f(x)$ to each number x in the domain. Purists are fond of emphasizing the distinction between the function f and its value $f(x)$ at x. However, once this distinction is clearly understood, most people who work with mathematics prefer to use the word loosely and speak of "the function $y = f(x)$," or even "the function $f(x)$."

The functions we work with in calculus are often composite (or compound) functions built up out of simpler ones. As an illustration of this idea, consider the two functions

$$f(x) = x^2 + 3x \qquad \text{and} \qquad g(x) = x^2 - 1.$$

The single function that results from first applying g to x and then applying f to $g(x)$ is

$$\begin{aligned} f(g(x)) &= f(x^2 - 1) = (x^2 - 1)^2 + 3(x^2 - 1) \\ &= x^4 + x^2 - 2. \end{aligned}$$

Notice that $f(x^2 - 1)$ is obtained by replacing x by the entire quantity $x^2 - 1$ in the formula $f(x) = x^2 + 3x$. The symbol $f(g(x))$ is read "f of g of x" and is called a *function of a function.* If we apply the functions in the other order (first f, then g), we have

$$\begin{aligned} g(f(x)) &= g(x^2 + 3x) = (x^2 + 3x)^2 - 1 \\ &= x^4 + 6x^3 + 9x^2 - 1, \end{aligned}$$

so $f(g(x))$ and $g(f(x))$ are different. In special cases it can happen that $f(g(x))$ and $g(f(x))$ are the same function of x; for example, if $f(x) = 2x - 3$ and $g(x) = -x + 6$:

$$\begin{aligned} f(g(x)) &= f(-x + 6) = 2(-x + 6) - 3 = -2x + 9, \\ g(f(x)) &= g(2x - 3) = -(2x - 3) + 6 = -2x + 9. \end{aligned}$$

In each of these examples two given functions are combined into a single composite function. In most practical work we proceed in the other direction, and dissect composite functions into their simpler constituents. For example, if

$$y = (x^3 + 1)^7,$$

we can introduce an auxiliary variable u by writing $u = x^3 + 1$ and decompose the above function into the two simpler functions

$$y = u^7 \qquad \text{and} \qquad u = x^3 + 1.$$

We shall see that decompositions of this kind are often useful in the problems of calculus.

In practice, functions often arise from algebraic relations between variables. Thus, an equation involving x and y determines y as a function of x if the equation is equivalent to one that expresses y *uniquely* in terms of x. For example, the equation $4x + 2y = 6$ can be solved for y, $y = 3 - 2x$, and this second equation defines y as a function of x. However, in some cases it happens that the process of solving for y leads to more than one value of y. For example, if the equation is $y^2 = x$, we get $y = \pm\sqrt{x}$. Since this gives two values of y for each positive value of x, the equation $y^2 = x$ does not by itself determine y as a function of x. If we wish, we can split the formula $y = \pm\sqrt{x}$ into two separate formulas, $y = \sqrt{x}$ and $y = -\sqrt{x}$. Each of these formulas defines y as a function of x, so that out of one equation we obtain two functions.

The number of distinct individual functions is clearly unlimited. However, most of those appearing in this book are relatively simple and can be classified into a few convenient categories. It may help students to orient themselves if we give a rough description of these categories in order of increasing complexity.

POLYNOMIALS

The simplest functions are the powers of x with nonnegative integer exponents,

$$1, x, x^2, x^3, \ldots, x^n, \ldots.$$

If a finite number of these are multiplied by constants and the results are added, we obtain a general polynomial,

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n.$$

The *degree* of a polynomial is the largest exponent that occurs in it; if $a_n \neq 0$, the degree of $p(x)$ is n. The following are polynomials of degrees 1, 2, and 3:

$$y = 3x - 2, \qquad y = 1 - 2x + x^2, \qquad y = x - x^3.$$

Polynomials can evidently be multiplied by constants, added, subtracted, and multiplied together, and the results are again polynomials.

RATIONAL FUNCTIONS

If division is also allowed, we pass beyond the polynomials into the more inclusive class of rational functions, such as

$$\frac{x}{x^2+1}, \qquad \frac{x+2}{x-2}, \qquad \frac{x^3-4x^2+x+6}{x^2+x+1}, \qquad x + \frac{1}{x}.$$

The general rational function is a quotient of polynomials,

$$\frac{a_0 + a_1x + a_2x^2 + \cdots + a_nx^n}{b_0 + b_1x + b_2x^2 + \cdots + b_mx^m},$$

and a specific function is rational if it is (or can be expressed as) such a quotient. If the denominator here is a nonzero constant, this quotient is itself a polynomial. Thus, the polynomials are included among the rational functions.

ALGEBRAIC FUNCTIONS

If root extractions are also allowed, we pass beyond the rational functions into the larger class of algebraic functions, which will be properly defined in a later chapter. Some simple examples are

$$y = \sqrt{x}, \qquad y = x + \sqrt[3]{x^2+1}, \qquad y = \frac{1}{\sqrt{1-x}}, \qquad y = \sqrt[4]{\frac{x+1}{x-1}}.$$

If we replace the root symbols by fractional exponents in accordance with the rules of algebra, then these functions can be written

$$y = x^{1/2}, \qquad y = x + (x^2+1)^{1/3}, \qquad y = (1-x)^{-1/2}, \qquad y = \left(\frac{x+1}{x-1}\right)^{1/4}.$$

TRANSCENDENTAL FUNCTIONS

Any function that is not algebraic is called *transcendental*. The transcendental functions studied in calculus are the trigonometric, inverse trigonometric, exponential, and logarithm functions. We do not assume that students have any previous knowledge of these functions. All will be carefully explained later.

We conclude this section with a brief review of some important functions arising in geometry. A ready grasp of the geometric formulas given in Fig. 1.22 is

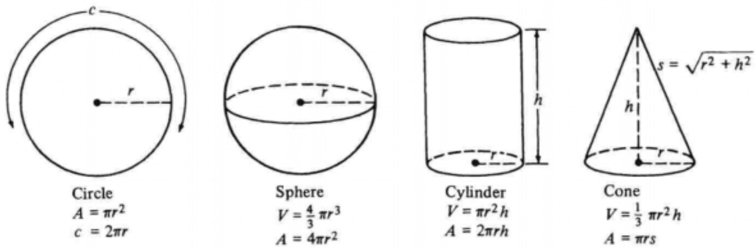

Figure 1.22 Geometric formulas.

essential for coping with many examples and problems in the following chapters. These formulas—for the area and circumference of a circle, the volume and total surface area of a sphere, and the volume and lateral surface area of a cylinder and a cone—should be understood if possible, but remembered in any event. Each of the first four formulas, those for the circle and the sphere, defines a function of the independent variable r, in which a given positive value of r determines the corresponding value of the dependent variable.

Most of our attention in this book will be directed at functions of a single independent variable, as previously defined and discussed. Nevertheless, we point out that each of the last four formulas in Fig. 1.22 defines a function of the two variables r and h; these variables are called *independent* (of each other) because the value assigned to either need not be related to the value assigned to the other. In special circumstances a function of this kind can be expressed as a function of one variable alone. For example, if the height of a cone is known to be twice the radius of its base so that $h = 2r$, then the formula for its volume can be written as a function of r or as a function of h:

$$V = \tfrac{1}{3}\pi r^2(2r) = \tfrac{2}{3}\pi r^3 \qquad \text{or} \qquad V = \tfrac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \tfrac{1}{12}\pi h^3.$$

The formulas in Fig. 1.22 also illustrate the custom of choosing letters for variables that suggest the quantities under discussion, such as A for area, V for volume, r for radius, h for height, and so on.

PROBLEMS

1 If $f(x) = 5x^2 - 3$, find:
(a) $f(-3)$; (b) $f(2)$;
(c) $f(0)$; (d) $f(-\sqrt{7})$;
(e) $f(a + 3)$; (f) $f(5t)$.

2 If $g(x) = \dfrac{x-1}{x+1}$, find:
(a) $g(3)$; (b) $g(-3)$;
(c) $g(\frac{1}{3})$; (d) $g\left(\dfrac{1}{a}\right)$;
(e) $g(a + 1)$; (f) $g(t - 1)$.

In each of Problems 3–8, compute and simplify the quantity $\dfrac{f(x+h) - f(x)}{h}$.

3 $f(x) = 5x - 3$. **4** $f(x) = 3 - 2x$.

5 $f(x) = x^2$. **6** $f(x) = 2x^2 + x$.

7 $f(x) = \dfrac{1}{x}$. **8** $f(x) = \dfrac{3}{1-x}$.

9 If $f(x) = x^3 - 3x^2 + 4x - 2$, compute $f(1)$, $f(2)$, $f(3)$, $f(0)$, $f(-1)$, and $f(-2)$.

10 If $f(x) = 2^x$, compute $f(1)$, $f(3)$, $f(5)$, $f(0)$, and $f(-2)$.

11 If $f(x) = 4x - 3$, show that $f(2x) = 2f(x) + 3$.

12 What are the domains of $f(x) = 1/(x - 8)$ and $g(x) = x^3$? What is $h(x) = f(g(x))$? What is the domain of $h(x)$?

13 Find the domain of each of the following functions:

(a) $\sqrt{x}$; (b) $\sqrt{-x}$;

(c) $\sqrt{x^2}$; (d) $\sqrt{x^2 - 4}$;

(e) $\dfrac{1}{x^2 - 4}$; (f) $\dfrac{1}{x^2 + 4}$;

(g) $\sqrt{(x - 1)(x + 2)}$; (h) $\dfrac{1}{\sqrt{(x - 1)(x + 2)}}$;

(i) $\sqrt{3 - 2x - x^2}$; (j) $\sqrt{\dfrac{x}{x - 2}}$.

14 If $f(x) = 1 - x$, show that $f(f(x)) = x$.

15 If $f(x) = x/(x - 1)$, compute $f(0)$, $f(1)$, $f(2)$, $f(3)$, and $f(f(3))$. Show that $f(f(x)) = x$.

16 If $f(x) = (ax + b)/(x - a)$, show that $f(f(x)) = x$.

17 If $f(x) = 1/(1 - x)$, compute $f(0)$, $f(1)$, $f(2)$, $f(f(2))$, and $f(f(f(2)))$. Show that $f(f(f(x))) = x$.

18 If $f(x) = ax$, show that $f(x) + f(1 - x) = f(1)$. Also verify that $f(x_1 + x_2) = f(x_1) + f(x_2)$ for all x_1 and x_2.

19 If $f(x) = 2^x$, use functional notation to express the fact that $2^{x_1} \cdot 2^{x_2} = 2^{x_1 + x_2}$.

20 Find $f(x)$ if $f(x + 1) = x^2 - 5x + 3$. Hint: Let $u = x + 1$ and find $f(u)$.

21 A *linear* function is one that has the form $f(x) = ax + b$, where a and b are constants. If $g(x) = cx + d$ is also linear, is it always true that $f(g(x)) = g(f(x))$?

22 If $f(x) = ax + b$ is a linear function with $a \neq 0$, show that there exists a linear function $g(x) = \alpha x + \beta$ such that $f(g(x)) = x$.* Also show that for these two functions it is true that $f(g(x)) = g(f(x))$.

23 A *quadratic* function is one that has the form $f(x) = ax^2 + bx + c$, where a, b, c, are constants and $a \neq 0$.

(a) Find the values of the coefficients a, b, c if $f(0) = 3$, $f(1) = 2$, $f(2) = 9$.

(b) Show that, no matter what values may be given to the coefficients, a, b, c, the range of a quadratic function cannot be the set of all real numbers.

24 In each case, decide whether or not the equation determines y as a function of x, and if it does, find a formula for the function:

(a) $3x^2 + y^2 = 1$; (b) $3x^2 + y = 1$;

(c) $\dfrac{y + 1}{y - 1} = x$; (d) $x = y - \dfrac{1}{y}$.

25 Split the equation $2x^2 + 2xy + y^2 = 3$ into two equations, each of which determines y as a function of x.

The following problems all involve geometry. In working on such a problem, always draw a sketch and use this sketch as a source of ideas.

26 If an equilateral triangle has side x, express its area as a function of x.

27 The equal sides of an isosceles triangle are 2. If x is the base, express the area as a function of x.

28 If the edge of a cube is x, express its volume, its surface area, and its diagonal as functions of x.

29 A rectangle whose base has length x is inscribed in a fixed circle of radius a. Express the area of the rectangle as a function of x.

30 A string of length L is cut into two pieces, and these pieces are shaped into a circle and a square. If x is the side of the square, express the total enclosed area as a function of x.

31 (a) Is the area of a circle a function of its circumference? If so, what function?

(b) Is the area of a square a function of its perimeter? If so, what function?

(c) Is the area of a triangle a function of its perimeter? If so, what function?

32 The volume of a sphere is a function of its surface area. Find a formula for this function.

33 A cylinder is inscribed in a sphere with fixed radius a. If h is the height and r is the radius of the base of the cylinder, express its volume and total surface area as functions of r, and also as functions of h.

34 A cylinder is circumscribed about a sphere. If their volumes are denoted by C and S, find C as a function of S.

35 A cylinder has fixed volume V. Express its total surface area as a function of the radius r of its base.

36 A fixed cone has height H and base radius R. If a cylinder with base radius r is inscribed in the cone, express the volume of the cylinder as a function of r.

37 (a) A farmer has 100 ft of fencing with which to build a rectangular chicken pen. If x is the length of one side of the pen, show that the enclosed area is

$$A = 50x - x^2 = 625 - (x - 25)^2.$$

Use this result to find the largest possible area and the lengths of the sides that yield this largest area.

(b) Suppose the farmer in part (a) decides to build the pen against a side of the barn so that he will have to fence only three sides of it. If x is the length of a side perpendicular to the barn wall, find the enclosed area as a function of x. Also find the largest possible area and the lengths of the sides that yield this largest area.

*The symbols α and β are letters of the Greek alphabet whose names are "alpha" and "beta." The letters of this alphabet (see the front endpaper) are used so frequently in mathematics and science that serious students should learn them at the earliest opportunity. Among other benefits, this will avoid the annoyance of reading printed matter containing symbols we don't know how to pronounce.

1.6 GRAPHS OF FUNCTIONS

In the previous section we discussed the concept of a function at some length. This discussion can be summarized in a few sentences, as follows.

If x and y are two variables that are related in such a way that whenever a suitable numerical value is assigned to x there is determined a single corresponding numerical value for y, then y is called *a function of* x and this is expressed by writing $y = f(x)$. The letter f symbolizes the function itself, which is the operation or rule of correspondence that yields y when applied to x. However, for practical reasons we prefer to speak of "the function $y = f(x)$" instead of "the function f." As a matter of principle, students should clearly understand that a function is not a formula and need not be specified by a formula—even though most of ours are.

Now for graphs.

The Chinese have a well-known proverb that can be interpreted as expressing a basic truth about the study of mathematics: One picture is worth a thousand words.* For us, in our study of functions, this means *draw graphs! Even more, cultivate the habit of thinking graphically, to the point where it becomes almost second nature.*

Before getting down to the details of specific functions, we emphasize that it is often possible to think of the graph of a function $y = f(x)$ very concretely, as the path of a moving point (Fig. 1.23). The independent variable x can be visualized as a point moving along the x-axis from left to right; each x determines a value of the dependent variable y, which is the height of the point (x, y) above the x-axis. The graph of the function is simply the path of the point (x, y) as it moves across the coordinate plane, sometimes rising and sometimes falling, and in general varying in height according to the nature of the particular function under consideration. The graph as a whole is intended to provide a clear overall picture of this variation. The graph shown in Fig. 1.23 happens to be a smooth curve with two high points and one low point, but many diverse phenomena are possible.

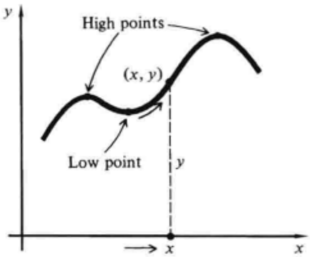

Figure 1.23

We now discuss the graphs of a few representative examples of the types of functions described in Section 1.5.

POLYNOMIALS

We have seen that the simplest polynomials are the powers of x with nonnegative integral exponents,

$$y = 1, x, x^2, x^3, \ldots, x^n, \ldots.$$

As we know, the graph of $y = 1$ is the horizontal straight line through the point $(0, 1)$, and the graph of $y = x$ is the straight line through the origin with slope 1 (Fig. 1.24*a*). For larger values of the exponent n, the graphs of $y = x^n$ are of two distinct types, depending on whether n is even or odd:

$$y = x^2, x^4, x^6, \ldots$$

and

$$y = x^3, x^5, x^7, \ldots.$$

*See *Bartlett's Familiar Quotations*, 16th ed. (Little, Brown and Co., 1992), fn. 8, p. 782.

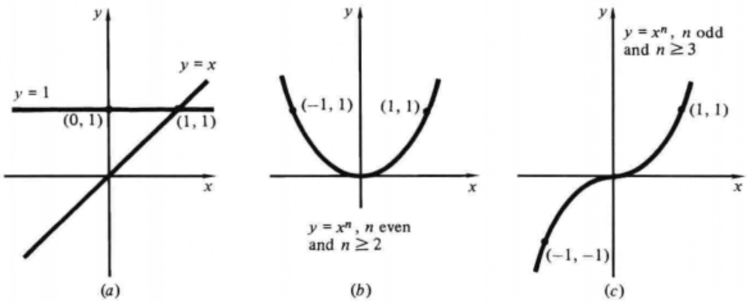

Figure 1.24 Graphs of $y = x^n$.

These types are shown in parts b and c of Fig. 1.24. As n increases, these curves become flatter near the origin and steeper outside the interval $[-1, 1]$.

We already know that the graphs of all first- and second-degree polynomials, such as

$$y = 2x - 1$$

and

$$y = 3x^2 - 2x + 1,$$

are straight lines and parabolas. These graphs are easy to draw—without plotting points—on the basis of the ideas in Sections 1.3 and 1.4.

For our next remark we need a bit of new terminology. A *zero* of a function $y = f(x)$ is a root of the corresponding equation $f(x) = 0$. Geometrically, the zeros of this function (if it has any) are the values of x at which its graph crosses or touches the x-axis; they are the x-intercepts of this graph.

Now consider the general second-degree polynomial

$$y = ax^2 + bx + c, \qquad a \neq 0. \tag{1}$$

As we know, the graph of this function is a parabola for all values of the coefficients. If we assume that $a > 0$, so that the parabola opens upward, then there are three possibilities for the zeros of (1), and these are shown in Fig. 1.25. Since the roots of the quadratic equation $ax^2 + bx + c = 0$ are given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

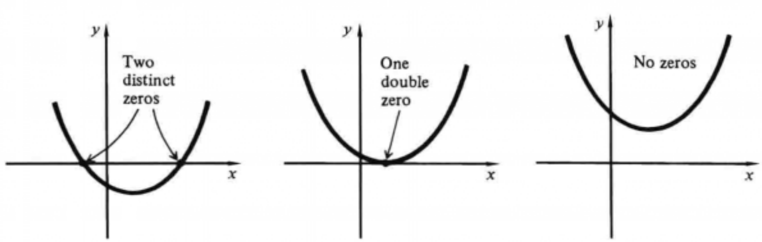

Figure 1.25

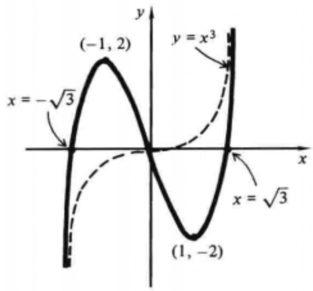

Figure 1.26

it is clear that the three possibilities in Fig. 1.25 correspond to the algebraic conditions $b^2 - 4ac > 0$, $b^2 - 4ac = 0$, $b^2 - 4ac < 0$.

The problem of graphing polynomials of degree $n \geq 3$ is not easy. Our discussion of the following example suggests several useful ideas.

Example 1 The graph of

$$y = x^3 - 3x \tag{2}$$

is shown in Fig. 1.26. At present we have no methods available for discovering such important features of this curve as the precise location of the indicated high and low points. This will come later. Nevertheless, a few observations can be made, and these provide at least some details and a good enough impression of the shape of the graph so that students should be able to sketch it for themselves.

We begin by pointing out that if (2) is written in factored form, as

$$y = x(x^2 - 3) = x(x + \sqrt{3})(x - \sqrt{3}), \tag{3}$$

Figure 1.27

then its zeros are obviously $0, -\sqrt{3}, \sqrt{3}$. These three numbers divide the x-axis into four intervals, as shown in Fig. 1.27, and a careful inspection of the factors of (3) tells us that in each interval y has the sign given in this figure. The details of this determination of the sign of y are important to understand, so we pause and carefully think it through, as follows:

for $x < -\sqrt{3}$, x is negative,
$x + \sqrt{3}$ is negative, and
$x - \sqrt{3}$ is negative,
so their product y is negative;

for $-\sqrt{3} < x < 0$, x is negative,
$x + \sqrt{3}$ is positive, and
$x - \sqrt{3}$ is negative,
so their product y is positive;

for $0 < x < \sqrt{3}$, x is positive,
$x + \sqrt{3}$ is positive, and
$x - \sqrt{3}$ is negative,
so their product y is negative;

for $x > \sqrt{3}$, x is positive,
$x + \sqrt{3}$ is positive, and
$x - \sqrt{3}$ is positive,
so their product y is positive.

We therefore know, for each interval, whether the graph of (2) lies above or below the x-axis (see Fig. 1.26). We have described this method of analysis in detail because it will often be useful in other problems of curve sketching.

Our second observation relates to the behavior of the graph of (2) when x is numerically large, that is, far to the right or far to the left in Fig. 1.26. If (2) is written in the form

$$y = x^3\left(1 - \frac{3}{x^2}\right), \qquad x \neq 0,$$

then for large positive or negative values of x the expression in parentheses is nearly 1, so y is close to x^3. In geometric language, when x is large, the graph of (2) is close to the graph of $y = x^3$, as Fig. 1.26 suggests. In particular, the graph of (2) rises on the far right and falls on the far left.

Students will notice that they can always sketch a graph by laboriously plotting many points and joining these points by a reasonable curve. Nevertheless, this rather clumsy procedure should be adopted only as a last resort, when more imaginative methods fail. The important features of functions and their graphs are much more clearly revealed by the qualitative approach to curve sketching that we have tried to suggest in Example 1 and will continue to emphasize.

RATIONAL FUNCTIONS

Example 2 The simplest rational function that is not a polynomial is

$$y = \frac{1}{x} \tag{4}$$

On examining (4), we notice the following facts: y is undefined when $x = 0$; y is positive when x is positive, and is small when x is large and large when x is near 0 on the right; y is negative when x is negative, and is small when x is large and large when x is near 0 on the left. The graph of (4) given in Fig. 1.28 is a direct pictorial version of these statements. In this particular case the graph is also easy to sketch by plotting a few points, as shown in the figure. However, students will profit much more from simply visualizing the behavior of such a function on the various parts of its domain and drawing what they see in the mind's eye.

A straight line is called an *asymptote* of a curve if, as a point moves out along an extremity of the curve, the distance from this point to the line approaches 0. It is clear that both the x-axis and the y-axis are asymptotes of the graph shown

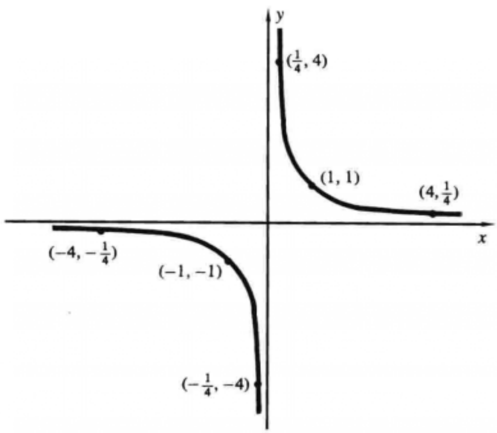

Figure 1.28

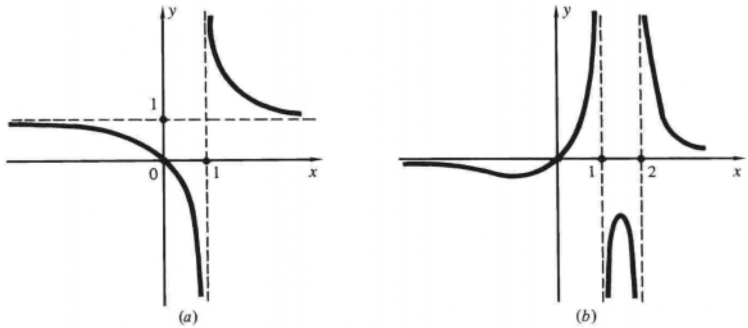

Figure 1.29

in Fig. 1.28. The behavior of the function (4) at and near the point $x = 0$, that is, the fact that y is undefined at $x = 0$ and "becomes infinite" near $x = 0$, is described by calling this point an *infinite discontinuity* of the function.

Example 3 In the case of the function

$$y = \frac{x}{x - 1}, \tag{5}$$

it is clear that the point $x = 1$ is particularly significant, since y is undefined at $x = 1$ and is large in absolute value when x is near 1 ($x = 1$ is an infinite discontinuity). Also, y is near 1 and slightly greater than 1 when x is large and positive, and is near 1 and slightly less than 1 when x is large and negative.* These observations suggest drawing the vertical and horizontal guidelines shown in Fig. 1.29*a*. If we notice that $y = 0$ when $x = 0$, and use the method of Example 1 to find the sign of y in each of the intervals $-\infty < x < 0$, $0 < x < 1$, and $1 < x$, then the graph as given in Fig. 1.29*a* is quite easy to sketch. The lines $x = 1$ and $y = 1$ are both asymptotes.

Example 4 The function

$$y = \frac{x}{x^2 - 3x + 2} = \frac{x}{(x - 1)(x - 2)} \tag{6}$$

is similar to (5) but somewhat more complicated. Here the factored form of the denominator reveals two infinite discontinuities, $x = 1$ and $x = 2$. Again, $y = 0$ when $x = 0$, but this time y is small when x is large, since the degree of the denominator is greater than that of the numerator. If we combine these facts with the observable sign of y in each of the intervals $-\infty < x < 0$, $0 < x < 1$, $1 < x < 2$, and $2 < x$ (think it through in the manner of Example 1 for each interval!), then it is fairly straightforward to sketch the graph as shown in Fig. 1.29*b*. There is evidently a high point between 1 and 2, and a low point to the left of 0, but at present we are unable to determine the precise location of these points (we shall see later that they occur at $x = \sqrt{2}$ and $x = -\sqrt{2}$).

---

*To see this, test with convenient specific values of x; thus, for example, $y = \frac{10}{9}$ when $x = 10$ and $y = \frac{10}{11}$ when $x = -10$.

Example 5 The function

$$y = x + \frac{1}{x} \tag{7}$$

has an infinite discontinuity at $x = 0$, and is positive or negative according as x is positive or negative. For small positive x's, the first term on the right of (7) is negligible and the second term is large; and for large positive x's, the second term is negligible and y is approximately equal to x. We therefore sketch the part of the graph in the right half-plane as follows: Draw the guideline $y = x$ (Fig. 1.30); insert the two extremities of the curve, approaching this guideline and the positive y-axis, as suggested by the behavior previously stated; and connect these extremities in a reasonable way in the middle, where this part of the graph has an obvious low point. The function behaves similarly—with a corresponding high point—for negative values of x. The y-axis and the line $y = x$ are both asymptotes.

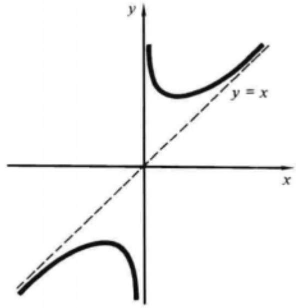

Figure 1.30

Example 6 The denominator of

$$y = \frac{x}{x^2 + 1} \tag{8}$$

is positive (in fact ≥ 1) for all x, so $y = 0$ when $x = 0$, y is positive when x is positive, and y is negative when x is negative. Also, y is small when x is large, because the degree of the denominator is greater than that of the numerator.* These properties of the function force the graph to have the shape shown in Fig. 1.31, with one high point and one low point.

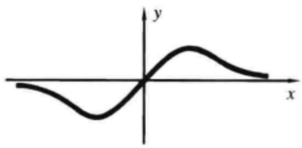

Figure 1.31

Example 7 In considering the function

$$y = \frac{x^2 - 1}{x - 1}, \tag{9}$$

it is natural to factor the numerator, obtaining

$$y = \frac{(x + 1)(x - 1)}{x - 1},$$

and then to cancel the common factor, which yields

$$y = x + 1.^{\dagger} \tag{10}$$

*Notice that when the numerator x is large, the denominator $x^2 + 1$ is enormous, so y is small.

†A word of warning about a point of algebra. To "cancel" a common *factor*, as in the text, is OK:

$$\frac{a\cancel{c}}{b\cancel{c}} = \frac{a}{b} \qquad \text{if } c \neq 0.$$

But "canceling" a common *term*, as in

$$\frac{a + \cancel{c}}{b + \cancel{c}} = \frac{a}{b},$$

is WRONG. Try it: Is

$$\frac{1 + 2}{2 + 2} = \frac{1}{2}?$$

Of course not.

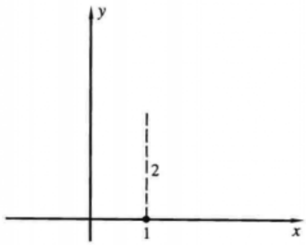

Figure 1.32

This cancellation is valid *except when* $x = 1$. At this point the value of (10) is 2, but (9) has no value ($y = 0/0$, which is meaningless). To graph (9), we therefore draw the straight line (10) and delete the single point (1, 2), as shown in Fig. 1.32.

Two functions $y = f(x)$ and $y = g(x)$ are said to be *equal* if they have the same domain and if $f(x) = g(x)$ for every x in their common domain. Accordingly, the functions (9) and (10) are not equal, because they have different domains—the point $x = 1$ is in the domain of (10) but is not in the domain of (9). The fact that the graph of (9) has a gap (or hole) corresponding to $x = 1$ is expressed by saying that (9) is *discontinuous* at $x = 1$, or has a *discontinuity* at this point.

ALGEBRAIC FUNCTIONS

Example 8 The functions

$$y = \sqrt{x} \qquad \text{and} \qquad y = \sqrt{25 - x^2} \tag{11}$$

can be obtained by solving the equations

$$y^2 = x \qquad \text{and} \qquad x^2 + y^2 = 25 \tag{12}$$

for y and choosing the positive square roots. We know that the graphs of equations (12) are a parabola and a circle, as shown in Fig. 1.33, so the graphs of (11) are the parts of these curves that lie on or above the x-axis.

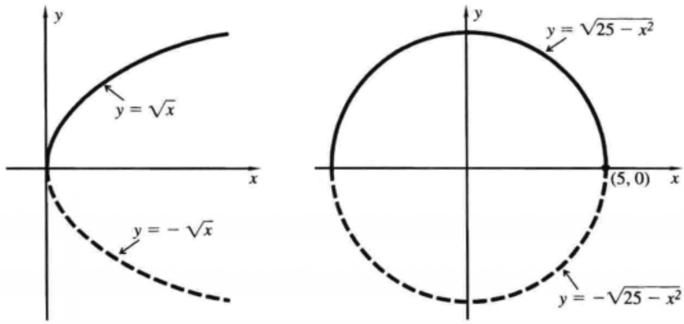

Figure 1.33

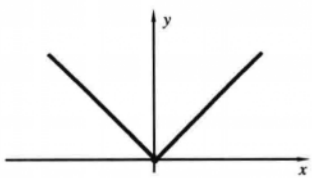

Figure 1.34

Example 9 The graph of the absolute value function

$$y = |x|$$

is easy to draw (Fig. 1.34). To see that this function is algebraic, we have only to notice the fact that $|x| = \sqrt{x^2}$ for every value of x.

As these examples show, many of the basic features of a function are made transparently clear by sketching its graph. We are interested less in sketches of high accuracy than in those that display broad general features: where the graph is rising and where falling, the presence of gaps, the presence of high points and low points, and what its approximate shape is. Formulas are obviously important in the study of functions—indeed, they are indispensable whenever our purposes require exact calculations yielding quantitative results. But we should never forget that *the primary aim of mathematics is insight,* and graphs are invaluable aids for gaining visual insight into the individual characteristics of functions.

PROBLEMS

1 Sketch the graphs of the following polynomials, paying special attention to the location of their zeros and their behavior for large values of x:
(a) $y = x^2 + x - 2$;
(b) $y = x^3 - 3x^2 + 2x$;
(c) $y = (1 - x)(2 - x)(3 - x)$;
(d) $y = x^4 - x^2$;
(e) $y = x^4 - 5x^2 + 4$.

2 Sketch the graphs of the following rational functions:
(a) $y = \dfrac{1}{x^2}$; (b) $y = \dfrac{1}{x^3}$;
(c) $y = x^2 + \dfrac{1}{x}$; (d) $y = x^2 + \dfrac{1}{x^2}$;
(e) $y = \dfrac{1}{x^2 + 1}$; (f) $y = \dfrac{x^2}{x^2 + 1}$;
(g) $y = \dfrac{1}{x^2 - 1}$; (h) $y = \dfrac{x}{x^2 - 1}$;
(i) $y = \dfrac{x^2}{x^2 - 1}$; (j) $y = \dfrac{x^2 - 3x + 2}{2 - x}$;
(k) $y = \dfrac{x^3 - x^2}{x - 1}$;
(l) $y = \dfrac{(x + 2)(x - 5)(x^2 + 2x - 8)}{(x - 2)(x^2 - 3x - 10)}$.

3 Sketch the graphs of the following algebraic functions:
(a) $y = \sqrt{(x - 1)(3 - x)}$;
(b) $y = \dfrac{1}{\sqrt{(x - 1)(3 - x)}}$;
(c) $y = \dfrac{1}{\sqrt{x - 1}}$; (d) $y = \sqrt{\dfrac{x}{3 - x}}$;
(e) $y = \sqrt{\dfrac{4 - x}{x - 2}}$; (f) $y = \sqrt{\dfrac{x - 4}{x - 2}}$.

4 In each of the following, sketch the graphs of all three functions on a single coordinate system:
(a) $y = |x|$, $y = |x| + 1$, $y = |x| - 1$;
(b) $y = |x|$, $y = |x + 1|$, $y = |x - 1|$;
(c) $y = |x|$, $y = 2|x|$, $y = \frac{1}{2}|x|$.

5 Sketch the graphs of the following functions:
(a) $y = \dfrac{|x|}{x}$; (b) $y = |2x + 3|$;
(c) $y = x + |x|$; (d) $y = 2x + |x|$;
(e) $y = x - |x|$; (f) $y = 1 + x - |x|$;
(g) $y = |x^2 - 1|$.

6 Considering only positive values of x, show that

$$y = \frac{|x + 1| - |x - 1|}{x} = \begin{cases} 2 & 0 < x < 1, \\ \dfrac{2}{x} & x \geq 1, \end{cases}$$

and sketch the graph.

7 Are any of the following pairs of functions equal?
(a) $f(x) = \dfrac{x}{x}$, $g(x) = 1$.
(b) $f(x) = x^2 - 1$, $g(x) = (x + 1)(x - 1)$.
(c) $f(x) = x$, $g(x) = \sqrt{x^2}$.
(d) $f(x) = x$, $g(x) = (\sqrt{x})^2$.

1.7 INTRODUCTORY TRIGONOMETRY. THE FUNCTIONS sin θ AND cos θ

Periodic phenomena are found everywhere in the world around us—vibrating springs, alternating currents, swinging pendulums, revolving planets, etc.—and scientists describe these phenomena by using trigonometric functions. For this and other reasons, students beginning the study of calculus are often expected to know something about trigonometry.

Although most users of this book have some familiarity with basic trigonometry, we nevertheless review a few of the fundamental ideas, especially the radian measure of angles and the definitions and simpler properties of the very im-

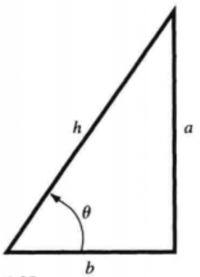

Figure 1.35

portant functions sin θ and cos θ.* This review is continued in Section 9.1, where the discussion is broadened to include the other four trigonometric functions tan θ, cot θ, sec θ, csc θ—all of which are indispensable in Chapter 10 but will not be needed until then.

In high school trigonometry courses the sine and cosine of an acute angle θ are first defined as ratios of sides in a right triangle, as follows (see Fig. 1.35):

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{a}{h},$$

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{b}{h}.$$

Because similar triangles have proportional sides, the values of sin θ and cos θ depend only on the size of the acute angle θ, and not at all on the size of the right triangle whose sides are used to compute these values.

Example 1 We know from geometry that in a 30°–60° right triangle, the side opposite the 30° angle is half the hypotenuse (see Problem 32 in Section 1.2). This enables us to draw the familiar right triangles shown in Fig. 1.36, and from these triangles we see that

$$\sin 30^\circ = \frac{1}{2}, \qquad \sin 60^\circ = \frac{\sqrt{3}}{2}, \qquad \sin 45^\circ = \frac{1}{\sqrt{2}},$$

$$\cos 30^\circ = \frac{\sqrt{3}}{2}, \qquad \cos 60^\circ = \frac{1}{2}, \qquad \cos 45^\circ = \frac{1}{\sqrt{2}}.$$

It is customary to rationalize the denominators on the right by writing

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{1}{2}\sqrt{2},$$

but for the moment we leave these values as they stand in order to emphasize the defining ratios.

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*The Greek letter θ is pronounced "theta."

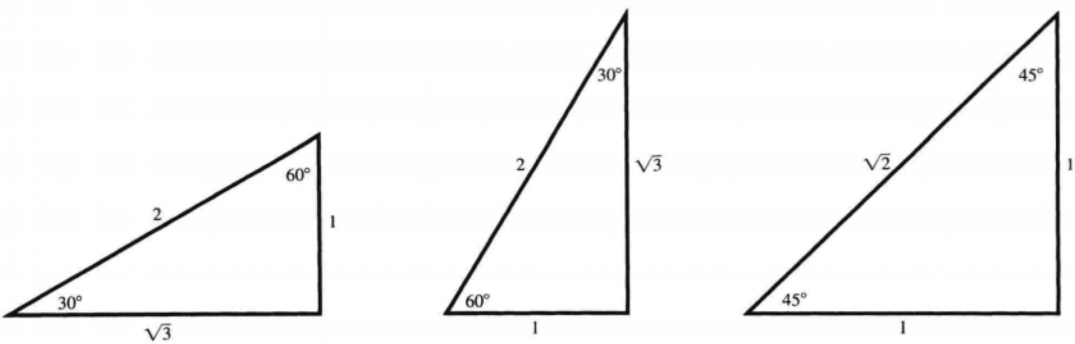


Figure 1.36

The ideas described here are part of what is called *right triangle trigonometry,* in which angles are measured in degrees and sines and cosines are defined only for acute angles of right triangles. In the equivalent forms

$$a = h\sin\theta \qquad \text{and} \qquad b = h\cos\theta,$$

these definitions have a number of applications in geometry and physics. This is all right as far as it goes. However, for the purposes of calculus the limitations of this approach are crippling. We therefore start all over again at the beginning and give a capsule development of *analytic trigonometry,* in which the trigonometric functions are freed from their dependence on right triangles and are defined as real-valued functions of a real variable. As an example of what we mean by analytic trigonometry, let us consider the motion of an object oscillating up and down at the end of a spring (Fig. 1.37). If this motion is described by the position function

$$s = f(t) = \cos t,$$

which gives the position s as a function of the time t, then it makes little sense to think of t as an angle and measure its values in degrees. We must consider what $\cos t$ means when t is not an angle but a *number*—the number of seconds that have elapsed since the motion began when $t = 0$.

Our treatment below is self-contained. Even a student who knows nothing of the subject will be able to learn everything that matters by reading with close attention and working through the problems at the end of the section.

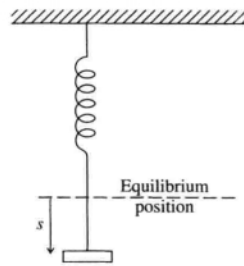
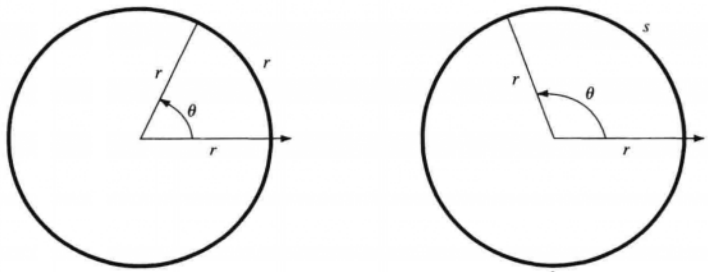


Figure 1.37

RADIAN MEASURE

In elementary mathematics and daily life, angles are measured in degrees, with 90° measuring a right angle. But the degree is an arbitrary measure inherited from the ancient Babylonian astronomers, and its use in calculus would make many of our formulas intolerably messy. In calculus we use a much more natural and convenient system called *radian measure,* which is defined in terms of how much arc an angle cuts off on a circle.

In this system the unit of angle measurement is called the *radian.* One radian is the angle which, placed at the center of a circle, subtends (cuts off) an arc whose length equals the radius (Fig. 1.38, left). More generally, the number of radians θ in an arbitrary central angle (Fig. 1.38, right) is defined to be the ratio of the length s of the subtended arc to the radius r, $\theta = s/r$, so that $s = r\theta$. We



$\theta = 1$ radian $\qquad\qquad \theta = \frac{s}{r}$ radians

Figure 1.38

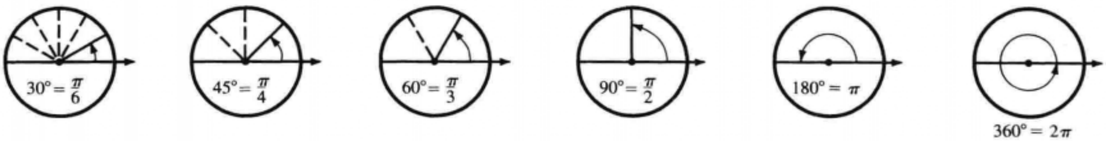


Figure 1.39

note especially that in a unit circle ($r = 1$), a central angle of θ radians subtends an arc of length $s = \theta$. Since the circumference of a circle is $c = 2\pi r$, a complete central angle of 360° is equivalent to $2\pi r/r = 2\pi$ radians. Thus,

$$2\pi \text{ radians} = 360° \qquad \text{or} \qquad \pi \text{ radians} = 180°;$$

and it follows from this that

$$1 \text{ radian} = \frac{180}{\pi} \cong 57.296°, \qquad 1° = \frac{\pi}{180} \cong 0.0175 \text{ radian}.$$

Further, $90° = \pi/2$, $60° = \pi/3$, $45° = \pi/4$, and $30° = \pi/6$, where we follow the convention of omitting the word "radian" in using radian measure. It is a good idea for students to memorize these common conversions with the aid of the circle diagrams in Fig. 1.39. In addition to *knowing* the conversions in these diagrams, it will help students feel more comfortable with radians if they also think through and verify the additional conversions in the following table.

Degrees	30	45	60	90	120	135	150	180	210	225	240	270	300	315	330	360
Radians	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π

The specific reason why radian measure for angles is preferred in calculus will appear in Section 3.4. In most of our work we will use radian measure routinely and mention degrees only in passing.

DEFINITIONS OF sin θ AND cos θ

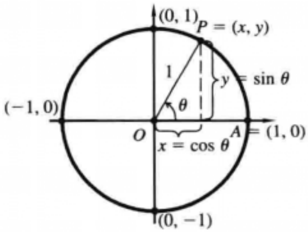


Figure 1.40

We approach trigonometry by way of analytic geometry. Consider the unit circle $x^2 + y^2 = 1$ in the xy-plane (Fig. 1.40), and let θ be an arbitrary real number. If θ is positive, let the radius OP start in the position OA and revolve counterclockwise through θ radians. Thus, $\theta = \pi$ produces half a revolution and $\theta = 2\pi$ produces a complete revolution, both counterclockwise. If θ is negative, we form the positive number $-\theta$ and let OP revolve clockwise through $-\theta$ radians. See Fig. 1.41. In this way, each real number θ (positive, negative, or zero) determines a unique position of the radius OP in Fig. 1.40, and therefore a unique point $P = (x, y)$ with the property that $x^2 + y^2 = 1$.

The *sine* and *cosine* of θ are now defined by

$$\sin\theta = y \qquad \text{and} \qquad \cos\theta = x.$$

The word "sine," *sinus* in Latin, is a corruption of an Arabic word meaning "chord" or "bowstring." Since sin and cos are the names of functions, the proper notation should be $\sin(\theta)$ and $\cos(\theta)$, just as we write $f(\theta)$ when the function is

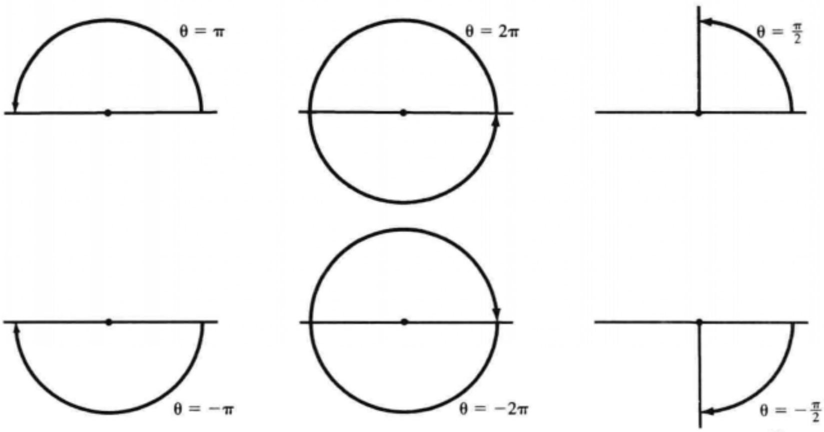


Figure 1.41

f. However, in the case of trigonometric functions it is customary to omit the parentheses. It is evident from the definition that $-1 \le \sin\theta \le 1$, and similarly for $\cos\theta$. The algebraic signs of these quantities depend on which quadrant of the plane the point P happens to lie in (Fig. 1.42). For values of θ such that $0 < \theta < \pi/2$, these definitions agree with the right triangle definitions given above, because in the triangle in Fig. 1.40 we have $\sin\theta = y = y/1 =$ (opposite side)/(hypotenuse) and $\cos\theta = x = x/1 =$ (adjacent side)/(hypotenuse).

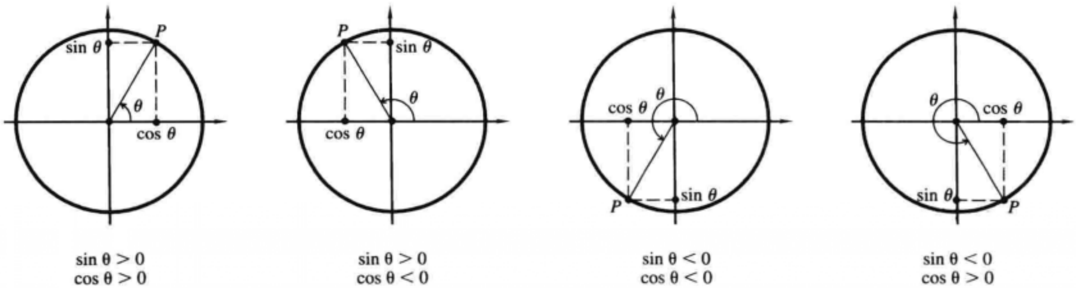


Figure 1.42

IDENTITIES

If we compare the angles θ and $-\theta$ in Fig. 1.43, we see at once that

$$\sin(-\theta) = -\sin\theta \quad \text{and} \quad \cos(-\theta) = \cos\theta. \tag{1}$$

The equation $x^2 + y^2 = 1$, or equivalently $y^2 + x^2 = 1$, translates immediately into the important identity

$$\sin^2\theta + \cos^2\theta = 1. \tag{2}$$

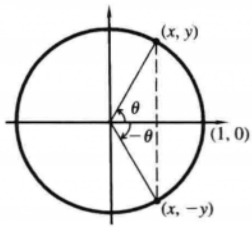


Figure 1.43

[The somewhat strange notation $\sin^2\theta$ is the standard way of writing the square of the number $\sin \theta$, that is, $(\sin \theta)^2$; and similarly for $\cos^2 \theta$.] Problem 10 in Section 9.1 outlines a general proof of the *addition formulas*

$$\sin(\theta + \phi) = \sin \theta \cos \phi + \cos \theta \sin \phi, \tag{3}$$

$$\cos(\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi. \tag{4}$$

We give a proof of (3) below, in connection with Fig. 1.44, for the restricted case in which θ and ϕ are both positive angles whose sum is less than $\pi/2$. First, however, we point out that if we put $\phi = \theta$ in (3) and (4), we obtain the *double-angle formulas*

$$\sin 2\theta = 2 \sin \theta \cos \theta, \tag{5}$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta. \tag{6}$$

And finally, if we write (2) and (6) together as

$$\cos^2 \theta + \sin^2 \theta = 1,$$

$$\cos^2 \theta - \sin^2 \theta = \cos 2\theta,$$

then by adding and subtracting we get the *half-angle formulas*

$$\cos^2 \theta = \tfrac{1}{2}(1 + \cos 2\theta), \tag{7}$$

$$\sin^2 \theta = \tfrac{1}{2}(1 - \cos 2\theta). \tag{8}$$

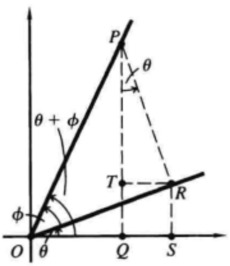


Figure 1.44

Now, to prove (3) for the restricted case mentioned above, we consult Fig. 1.44 and write

$$\begin{aligned}
\sin(\theta + \phi) &= \frac{PQ}{OP} = \frac{PT + TQ}{OP} \\
&= \frac{PT + RS}{OP} = \frac{PT}{OP} + \frac{RS}{OP} \\
&= \frac{PT}{PR} \cdot \frac{PR}{OP} + \frac{RS}{OR} \cdot \frac{OR}{OP} \\
&= \cos \theta \sin \phi + \sin \theta \cos \phi.
\end{aligned}$$

A similar argument can be given for formula (4).

VALUES AND GRAPHS

Example 1 provides several first-quadrant θ's for which exact values of $\sin \theta$ and $\cos \theta$ are easy to find. These facts can also be obtained by looking carefully at the three parts of Fig. 1.45 and remembering the Pythagorean theorem:

$$\sin \frac{\pi}{6} = \frac{1}{2}, \qquad \sin \frac{\pi}{4} = \frac{1}{2}\sqrt{2}, \qquad \sin \frac{\pi}{3} = \frac{1}{2}\sqrt{3},$$

$$\cos \frac{\pi}{6} = \frac{1}{2}\sqrt{3}, \qquad \cos \frac{\pi}{4} = \frac{1}{2}\sqrt{2}, \qquad \cos \frac{\pi}{3} = \frac{1}{2}.$$

Also, an inspection of Fig. 1.40 with OP in various positions gives us similar information for the cases $\theta = 0, \pi/2, \pi, 3\pi/2, 2\pi$:

$$\sin 0 = 0, \qquad \sin \frac{\pi}{2} = 1, \qquad \sin \pi = 0, \qquad \sin \frac{3\pi}{2} = -1, \qquad \sin 2\pi = 0,$$

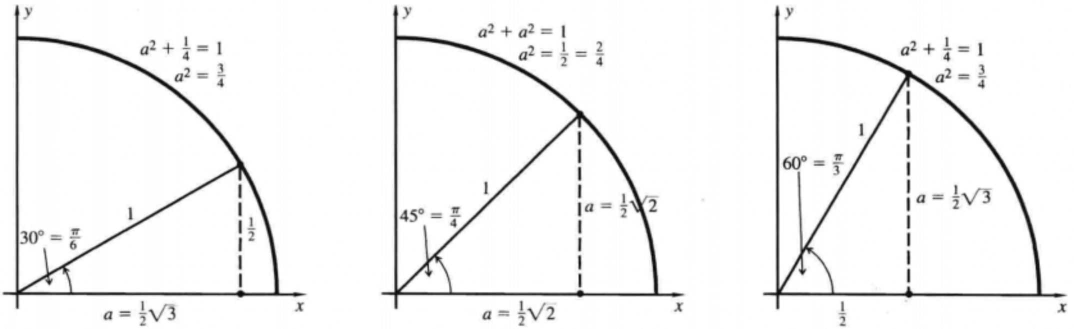


Figure 1.45

$$\cos 0 = 1, \qquad \cos\frac{\pi}{2} = 0, \qquad \cos \pi = -1, \qquad \cos\frac{3\pi}{2} = 0, \qquad \cos 2\pi = 1.$$

Further, by drawing pictures and using the ideas in Fig. 1.45 we can find the exact values of sin θ and cos θ for any value of θ that represents an angle one-third, one-half, or two-thirds of the way through any quadrant.

Example 2 To illustrate this remark, we point out (Fig. 1.46) that $135° = 3\pi/4$ is halfway from $\pi/2$ to π, so the point P is in the second quadrant with coordinates $(-\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2})$, and consequently we have

$$\sin\frac{3\pi}{4} = \frac{1}{2}\sqrt{2}, \qquad \cos\frac{3\pi}{4} = -\frac{1}{2}\sqrt{2}.$$

Similarly, $300° = 5\pi/3$ is one-third of the way from $3\pi/2$ to 2π, so P is in the fourth quadrant with coordinates $(\frac{1}{2}, -\frac{1}{2}\sqrt{3})$, and we have

$$\sin\frac{5\pi}{3} = -\frac{1}{2}\sqrt{3}, \qquad \cos\frac{5\pi}{3} = \frac{1}{2}.$$

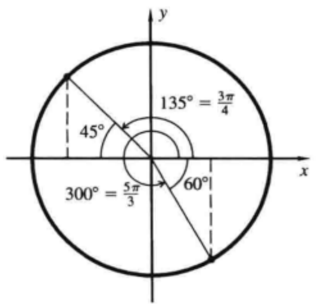


Figure 1.46

Of course, most θ's are beyond the scope of these methods, and in these cases the values of sin θ and cos θ can be found from trigonometric tables or a calculator. The problem of how these values themselves are calculated is more difficult, and will be discussed in Chapter 14.

For every θ, the numbers θ and $\theta + 2\pi$ clearly determine the same point P, so

$$\sin(\theta + 2\pi) = \sin\theta \qquad \text{and} \qquad \cos(\theta + 2\pi) = \cos\theta.$$

This says that the values of sin θ and cos θ repeat when θ increases by 2π. We express these properties of sin θ and cos θ by saying that these functions are *periodic* with *period* 2π.

The graph of sin θ is easy to sketch by looking at Fig. 1.40 and using imagination to follow the way y varies as θ increases from 0 to 2π, that is, as the radius swings around through one complete counterclockwise revolution. It is clear that sin θ starts at 0, increases to 1, decreases to 0, decreases further to -1, and increases to 0. This gives one complete cycle of sin θ on the interval $0 \le \theta \le 2\pi$, as shown on the left in Fig. 1.47. By using the periodicity of sin θ, we see

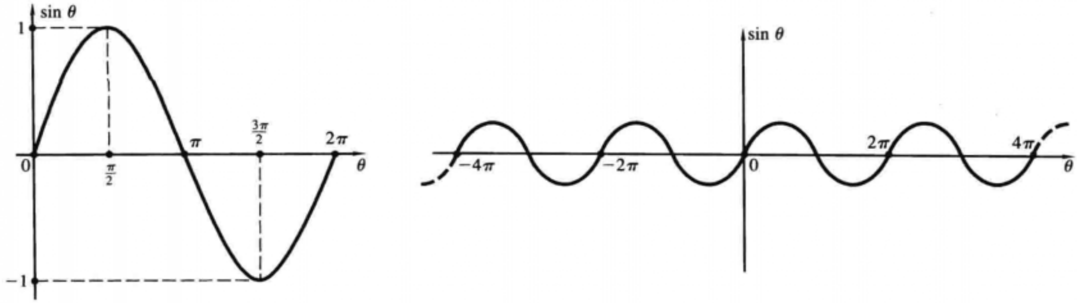


Figure 1.47

that the complete graph (on the right in the figure) consists of infinitely many repetitions of this cycle, to the right and to the left. The graph of $\cos\theta$ can be sketched in essentially the same way (Fig. 1.48). The main difference is that $\cos\theta$ starts at 1 when $\theta = 0$, decreases to 0, decreases further to -1, increases to 0, and increases further to 1.

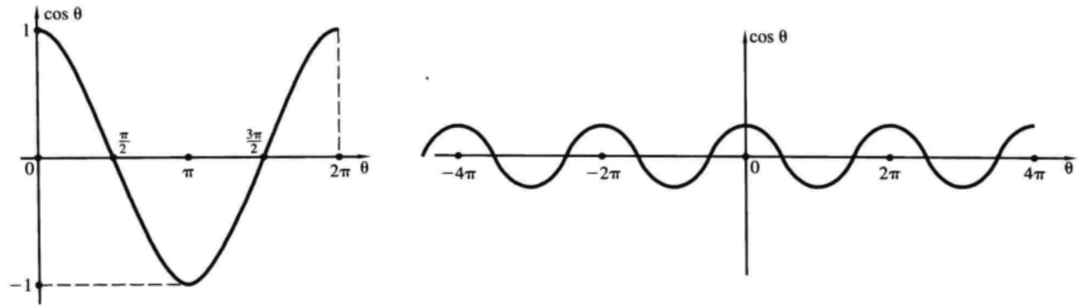


Figure 1.48

On the other hand, the graph of $\sin 2\theta$ makes one complete cycle on the interval $0 \leq \theta \leq \pi$, because 2θ increases from 0 to 2π as θ increases from 0 to π (Fig. 1.49, left). This says that $\sin 2\theta$ oscillates twice as fast as $\sin\theta$. In the same

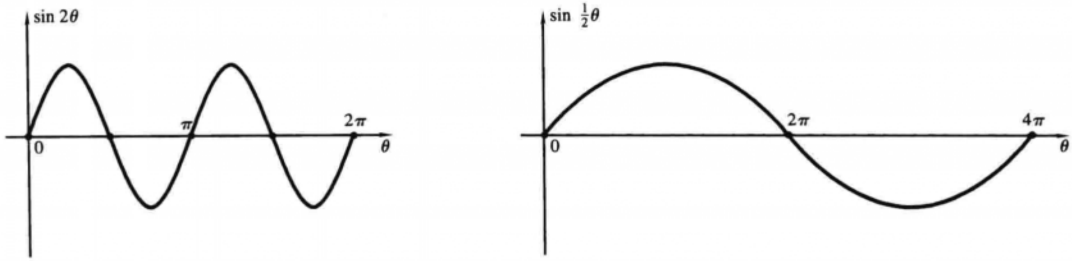


Figure 1.49

way we see that $\sin \frac{1}{2}\theta$ oscillates half as fast as $\sin\theta$ (Fig. 1.49, right). In general, both $\sin k\theta$ and $\cos k\theta$ make one complete cycle for $0 \le k\theta \le 2\pi$, or equivalently, on the interval $0 \le \theta \le 2\pi/k$.

Notice that degrees are almost entirely banished from this way of thinking about trigonometry. Trigonometric *values* can be written using degree measure or radian measure: either $\sin 30°$ or $\sin \pi/6$; either $\cos 90°$ or $\cos \pi/2$. But whenever we think of trigonometric *functions*, as in writing $y = \sin\theta$ or $f(\theta) = \cos\theta$, the independent variable θ is always understood to be in radians.

The functions $\sin\theta$ and $\cos\theta$ are the basic trigonometric functions, but there are four others that are also important though less fundamental: the tangent, cotangent, secant, and cosecant. These can be defined as follows:

$$\tan\theta = \frac{\sin\theta}{\cos\theta},$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta},$$

$$\sec\theta = \frac{1}{\cos\theta},$$

$$\csc\theta = \frac{1}{\sin\theta}.$$

Even though we mention them here, these four functions will not be essential for our work until we reach Chapter 10. At that time we will review them thoroughly.

PROBLEMS

1 Convert the given angle from degrees to radians:
(a) $15°$; (b) $150°$;
(c) $1500°$; (d) $-36°$;
(e) $-110°$; (f) $7°$.

2 Convert the given angle from radians to degrees:
(a) $\pi/15$; (b) $\pi/45$;
(c) $-\pi/36$; (d) -3;
(e) π^2; (f) 30.

3 Find the value of the given expression without using tables or a calculator:
(a) $\cos(-120°)$; (b) $\sin 780°$;
(c) $\sin \frac{17\pi}{3}$; (d) $\cos\left(-\frac{15\pi}{4}\right)$;
(e) $\sin \frac{19\pi}{6}$; (f) $\cos \frac{99\pi}{4}$.

4 Is the given number positive, negative, or zero?
(a) $\sin 500\pi$; (b) $\cos 7$;
(c) $\sin 901°$; (d) $\cos 2^4$.

5 Verify the given identities:
(a) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ (Hint: $3\theta = 2\theta + \theta$);
(b) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

6 By sketching the angles in a unit circle and using the facts that $\sin \pi/6 = \frac{1}{2}$, $\cos \pi/6 = \sqrt{3}/2$, find
(a) $\sin\left(-\frac{\pi}{6}\right)$; (b) $\sin \frac{7\pi}{6}$;
(c) $\sin \frac{13\pi}{6}$; (d) $\cos\left(-\frac{\pi}{6}\right)$;
(e) $\cos \frac{7\pi}{6}$; (f) $\cos \frac{13\pi}{6}$.

7 Express each trigonometric function as a corresponding function of an angle in the first quadrant ($0 \le \theta \le \pi/2$) preceded by a + or − sign:
(a) $\sin \frac{9\pi}{2}$; (b) $\sin 7\pi$;
(c) $\sin\left(-\frac{7\pi}{3}\right)$; (d) $\sin\left(-\frac{8\pi}{3}\right)$;
(e) $\cos 10\pi$; (f) $\cos \frac{9\pi}{4}$;
(g) $\cos\left(-\frac{6\pi}{5}\right)$; (h) $\sin\left(-\frac{11\pi}{2}\right)$;
(i) $\cos \frac{11\pi}{3}$.

8 Replace ϕ by $-\phi$ in the addition formulas (3) and (4), and use the identities (1), to obtain the *subtraction formulas*:

$$\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi,$$
$$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi.$$

9 By examining Fig. 1.50, obtain the following identities:
(a) $\sin(\pi - \theta) = \sin\theta$, $\cos(\pi - \theta) = -\cos\theta$;
(b) $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$, $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$.
Use similar arguments—based on appropriate pictures—to obtain identities (c) and (d):
(c) $\sin(\theta + \pi) = -\sin\theta$, $\cos(\theta + \pi) = -\cos\theta$;
(d) $\sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta$, $\cos\left(\theta + \frac{\pi}{2}\right) = -\sin\theta$.

10 Derive the identities in Problem 9 as special cases of the addition and subtraction formulas.

11 The half-angle formulas (7) and (8) are called by this name because if we set $2\theta = \alpha$, they can be written as

$$\sin\tfrac{1}{2}\alpha = \pm\sqrt{\frac{1 - \cos\alpha}{2}},$$
$$\cos\tfrac{1}{2}\alpha = \pm\sqrt{\frac{1 + \cos\alpha}{2}}.$$

Use these formulas to find the values of $\sin 15°$ and $\cos 15°$.

12 Apply the formulas in Problem 11 to find the values of $\sin 30°$ and $\cos 30°$ from the fact that $\cos 60° = \frac{1}{2}$.

13 Apply the half-angle formula for the cosine to find
(a) $\cos\frac{\pi}{4}$; (b) $\cos\frac{3\pi}{4}$.

14 Apply the half-angle formula for the sine to find
(a) $\sin\frac{\pi}{4}$; (b) $\sin\left(-\frac{\pi}{2}\right)$.

15 Use the appropriate addition or subtraction formula to find
(a) $\sin\frac{2\pi}{3}$ from $\frac{2\pi}{3} = \pi - \frac{\pi}{3}$;
(b) $\cos\frac{5\pi}{4}$ from $\frac{5\pi}{4} = \pi + \frac{\pi}{4}$;
(c) $\sin\frac{17\pi}{6}$ from $\frac{17\pi}{6} = 3\pi - \frac{\pi}{6}$.

16 Use the method of the preceding problem to find
(a) $\cos\frac{19\pi}{6}$; (b) $\cos\frac{10\pi}{3}$; (c) $\sin\frac{11\pi}{6}$.

17 Check the identity for $\sin(\theta + \phi)$ when
(a) $\theta = \frac{\pi}{6}$ and $\phi = \frac{\pi}{3}$;
(b) $\theta = \frac{\pi}{4}$ and $\phi = \frac{\pi}{4}$.

18 Check the identity for $\cos(\theta + \phi)$ when
(a) $\theta = \frac{\pi}{6}$ and $\phi = \frac{\pi}{3}$;
(b) $\theta = \frac{\pi}{4}$ and $\phi = \frac{\pi}{4}$.

19 Find $\sin 5\pi/12$ by using the fact that $5\pi/12 = \pi/4 + \pi/6$.

20 Find $\sin \pi/12$ by using the fact that $\pi/12 = \pi/4 - \pi/6$. Reconcile your answer here with the first answer in Problem 11.

21 Establish the addition formula (4) for the cosine by the method suggested in the text, that is, by using Fig. 1.44.

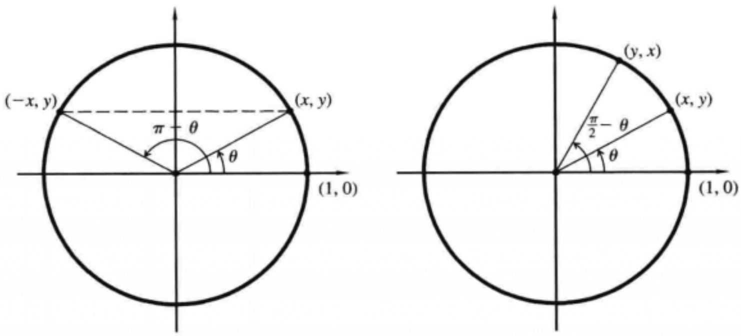


Figure 1.50

CHAPTER 1 REVIEW: DEFINITIONS, CONCEPTS, METHODS

Define, state, or think through the following.

1 Rational and irrational numbers.
2 Real line.
3 Rules for inequalities.
4 Absolute value of a number.
5 Closed and open intervals.
6 Coordinate plane.
7 Pythagorean theorem.

8 Distance formula.
9 Midpoint formulas.
10 Slope of a straight line.
11 Point-slope equation of a line.
12 Slope-intercept equation of a line.
13 Slope criterion for parallel lines.
14 Slope criterion for perpendicular lines.
15 Equation of a circle.
16 Completing the square.
17 Definition of a parabola.
18 Equations of parabolas.
19 Function.
20 Domain and range of a function.
21 Independent and dependent variables.
22 Polynomials.
23 Rational functions.
24 Algebraic functions.
25 Transcendental functions.
26 Graph of a function.
27 Zero of a function.
28 Asymptote of a curve.
29 Infinite discontinuity of a function.
30 Radian measure.
31 Sine and cosine of θ.
32 Addition and subtraction formulas.
33 Values and graphs of $\sin\theta$ and $\cos\theta$.
34 Double-angle formulas.
35 Half-angle formulas.

ADDITIONAL PROBLEMS FOR CHAPTER 1

SECTION 1.2

1 If a and b are positive numbers, prove the inequality $\sqrt{ab} \le \frac{1}{2}(a+b)$ as Euclid did, by considering a right triangle inscribed in a semicircle (Fig. 1.51).

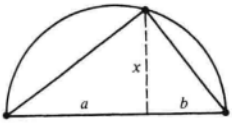


Figure 1.51

2 If a and b are any two numbers, denote the larger by max (a, b) and the smaller by min (a, b). Show that

$$\max(a, b) = \tfrac{1}{2}(a + b + |a - b|),$$

and find a similar expression for min (a, b).

3 Show that if $a \le b$ and $c \le d$, then $a + c \le b + d$. Use this fact to prove that $|a + b| \le |a| + |b|$. Hint: Begin by noticing that $-|a| \le a \le |a|$ and $-|b| \le b \le |b|$.

4 If a is a positive rational number, explain why the following method for calculating the square root of a works. First, choose a rational number which is a reasonable guess at the value of $\sqrt{a}$, and call this initial approximation x_1. Next, divide a by x_1 and average the result with x_1, thereby obtaining a second approximation x_2. Next, divide a by x_2 and average the result with x_2, obtaining a third approximation x_3. This procedure is expressed by the formula

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right), \qquad n = 1, 2, 3, \ldots .$$

Hint: If x_1 is reasonably close to $\sqrt{a}$ but different from it, then $\sqrt{a}$ lies between x_1 and a/x_1 (why?), and so the average of x_1 and a/x_1 is likely to be even closer to $\sqrt{a}$; also note that

$$x_{n+1} - \sqrt{a} = \frac{1}{2}\left(x_n - 2\sqrt{a} + \frac{a}{x_n}\right) = \frac{1}{2x_n}(x_n - \sqrt{a})^2.$$

5 Use the method of Problem 4 to calculate $\sqrt{2}$, first with $x_1 = 1$ and then with $x_1 = \frac{3}{2}$.

6 Use the method of Problem 4 to calculate $\sqrt{3}$, first with $x_1 = 2$ and then with $x_1 = \frac{3}{2}$.

7 If a and b are real numbers with $a < b$, show that there exists at least one rational number c such that $a < c < b$, and hence infinitely many. In particular, between any two irrationals there exist an infinite number of rationals.

8 If a is a nonzero rational number and b is irrational, show that $a + b$, $a - b$, ab, a/b, and b/a are all irrational.

9 If a and b are irrational, is $a + b$ necessarily irrational? Is ab?

10 If a and b are real numbers with $a < b$, show that there exists at least one irrational number c such that $a < c < b$, and hence infinitely many. In particular, between any two rationals there exist an infinite number of irrationals.

11 Give another proof of the Pythagorean theorem by using the equations

$$\frac{a}{c} = \frac{e}{a} \qquad \text{and} \qquad \frac{b}{c} = \frac{d}{b},$$

obtained from similar triangles in Fig. 1.52.*

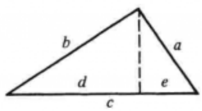


Figure 1.52

*Further proofs can be found in Section B.1 of Simmons, *Calculus Gems* (McGraw-Hill, 1992).

12 In each case place the figure in a convenient position relative to the coordinate system and prove the statement algebraically:
(a) The sum of the squares of the distances of any point from two opposite vertices of a rectangle equals the sum of the squares of its distances from the other two vertices.
(b) In any triangle, 4 times the sum of the squares of the medians equals 3 times the sum of the squares of the sides.

13 If $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ are distinct points, and if $P = (x, y)$ is located on the segment joining them in such a position that the ratio of its distance from P_1 to its distance from P_2 is q/p, show that

$$x = \frac{px_1 + qx_2}{p + q} \quad \text{and} \quad y = \frac{py_1 + qy_2}{p + q}.$$

14 Find the point on the segment joining (1, 2) and (5, 9) that is $\frac{11}{17}$ of the way from the first point to the second.

SECTION 1.3

15 If the line determined by two distinct points (x_1, y_1) and (x_2, y_2) is not vertical, and therefore has slope $(y_2 - y_1)/(x_2 - x_1)$, show that the point-slope form of its equation is the same regardless of which point is used as the given point.

16 Determine what each of the following statements implies about the constants A, B, C in the equation $Ax + By + C = 0$:
(a) The line goes through the origin.
(b) The line is parallel to the y-axis.
(c) The line is perpendicular to the y-axis.
(d) The line goes through (1, 1).
(e) The line is parallel to $5x + 3y = 2$.
(f) The line is perpendicular to $x + 10y = 3$.

17 If the lines $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ are not parallel and k is any constant, show that

$$(A_1x + B_1y + C_1) + k\,(A_2x + B_2y + C_2) = 0$$

is a line through the point of intersection of the given lines. When k is assigned various values, this equation represents various members of the family of all lines through the point of intersection.

18 Given the lines $x + 3y - 2 = 0$ and $2x - y + 4 = 0$, use Problem 17 to find the equation of the line through their point of intersection which
(a) passes through $(-2, 1)$;
(b) is perpendicular to the line $3y + x = 21$;
(c) passes through the origin.

19 The points (0, 0), $(a, 0)$, and (b, c) are the vertices of an arbitrary triangle which is placed in a convenient position relative to the coordinate system.
(a) Find the equation of the line through each vertex perpendicular to the opposite side, and show algebraically that these three lines intersect at a single point.
(b) Find the equation of the perpendicular bisector of each side, and show algebraically that these three lines intersect at a single point. Why is this fact geometrically obvious?
(c) Find the equation of the line through each vertex and the midpoint of the opposite side, and show algebraically that these three lines intersect at a single point. Also, verify that this point is two-thirds of the way from each vertex to the midpoint of the opposite side.

20 Show that each of the following is the equation of a straight line:
(a) $x^3 - x^2y - 2x^2 + 3x - 3y - 6 = 0$.
(b) $3xy^2 + 5y^2 - y^3 - 4y + 12x + 20 = 0$.

21 Show that the distance from a point (x_0, y_0) to a line $Ax + By + C = 0$ is given by

$$\frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}.$$

22 Find the distance between the parallel lines $4x + 3y + 12 = 0$ and $4x + 3y - 38 = 0$.

23 If two intersecting straight lines are given, then it is easy to see that the bisectors of the angles formed by these lines are two other straight lines whose points are equidistant from the given lines. Use this fact to find the equations of the bisectors of the angles formed by the lines
(a) $3x + 4y - 10 = 0$ and $4x - 3y - 5 = 0$;
(b) $y = 0$ and $y = x$.

24 Why is it geometrically obvious (without calculation) that the bisectors of the angles of any triangle intersect at a single point?

SECTION 1.4

25 Find the values of b for which the line $y = 3x + b$ intersects the circle $x^2 + y^2 = 4$.

26 If the line $y = mx + b$ is tangent to the circle $x^2 + y^2 = r^2$, find an equation relating m, b, and r.

27 Find the equation of the locus of a point $P = (x, y)$ that moves in such a way that
(a) its distance from (0, 0) is twice its distance from $(a, 0)$;
(b) the product of its distances from $(a, 0)$ and $(-a, 0)$ is a^2 (this curve is called a *lemniscate*).
In each case, sketch the graph.

28 A line segment of length 6 moves in such a way that its endpoints remain on the x-axis and y-axis. What is the equation of the locus of its midpoint?

29 A point moves in such a way that the ratio of its distances from two fixed points is a constant $k \neq 1$. Show that the locus is a circle.

30 Find the equation of the line which is tangent to the circle $x^2 + y^2 + 8x + 6y + 8 = 0$ at the point $(-8, -2)$.

31 Find the equations of the lines that pass through the point $(1, 3)$ and are tangent to the circle $x^2 + y^2 = 2$.

32 If two circles

$$x^2 + y^2 + A_1x + B_1y + C_1 = 0$$

and

$$x^2 + y^2 + A_2x + B_2y + C_2 = 0$$

intersect in two points, and if k is a constant $\neq -1$, explain why

$$(x^2 + y^2 + A_1x + B_1y + C_1) + k(x^2 + y^2 + A_2x + B_2y + C_2) = 0$$

is the equation of a circle through the points of intersection. If $k = -1$, what does the equation represent?

33 Use Problem 32 to find the equation of the line joining the points of intersection of the circles $x^2 + y^2 = 4x + 4y - 4$ and $x^2 + y^2 = 2y$. Also find these points of intersection.

34 Show that a parabola with focus at the origin, axis the x-axis, and opening to the right has an equation of the form $y^2 = 4p(x + p)$, where $p > 0$.

35 Find the equation of the parabola with focus $(1, 1)$ and directrix $x + y = 0$, and simplify this equation to a form without radicals. Hint: See Problem 21.

36 Let the vertex of the parabola $x^2 = 4py$ be joined to every other point of the parabola. Show that the midpoints of the resulting chords lie on another parabola. Find the focus and directrix of this second parabola.

37 Consider all chords with given slope m that have endpoints on the parabola $x^2 = 4py$. Prove that the locus of the midpoints of these chords is a straight line parallel to the y-axis.

38 A *focal chord* of a parabola is the segment cut by the parabola from a straight line through the focus.

(a) If A and B are the endpoints of a focal chord, and if the line through A and the vertex intersects the directrix at a point C, show that the line through B and C is parallel to the axis of the parabola.

(b) Show that the length of a focal chord is twice the distance from its midpoint to the directrix.

(c) Show that if the two tangents to a parabola are drawn from any point on the directrix, then the points of tangency are the endpoints of a focal chord.

39 Given the two points $A = (4p, 0)$ and $B = (4p, 4p)$, divide the segments OA and AB into equal numbers of equal parts, number the points of division as shown in Fig. 1.53, and join the points of division on AB to the origin by straight lines. Show that the points of intersection of each of these lines with the corresponding vertical lines lie on the parabola $x^2 = 4py$.

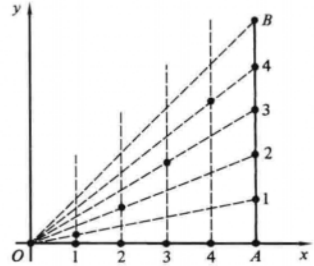


Figure 1.53

SECTION 1.5

40 Find the domain of each of the following functions:

(a) $5 - x$; (b) $\dfrac{x}{2x - 3}$;

(c) $\sqrt{3x - 2}$; (d) $\sqrt{5 - 3x}$;

(e) $\dfrac{x + 7}{x^2 - 9}$; (f) $\sqrt[3]{x}$;

(g) $\sqrt{9 - 4x^2}$; (h) $\dfrac{1}{\sqrt{x + 3}}$;

(i) $\sqrt{7x^2 + 5}$.

41 If $f(x) = ax + b$, show that

$$f\left(\frac{x_1 + x_2}{2}\right) = \frac{f(x_1) + f(x_2)}{2}.$$

Is this true for $f(x) = x^2$?

42 If $f(x) = (1 + x)/(1 - x)$, find

(a) $f(-x)$; (b) $f\left(\dfrac{1}{x}\right)$;

(c) $f\left(\dfrac{1}{1 - x}\right)$; (d) $f(f(x))$.

43 If $f(x) = \sqrt[3]{x}$, what function $g(x)$ has the property that $g(f(x)) = x$?

44 The perimeter of a right triangle is 6 and its hypotenuse is x. Express the area as a function of x.

45 A cylinder has fixed total surface area A. Express its volume as a function of the radius r of its base.

46 A cone is inscribed in a sphere with fixed radius a. If r is the radius of the base of the cone, express its volume as a function of r.

47 A cone is circumscribed about a sphere with fixed radius a. If r is the radius of the base of the cone, express its volume as a function of r.

48 If $f(x) = (x - 3)/(x + 1)$, show that $f(f(f(x))) = x$.

49 Let a, b, c, d be given constants with the property that $ad - bc \neq 0$. If $f(x) = (ax + b)/(cx + d)$, show that there exists a function $g(x) = (\alpha x + \beta)/(\gamma x + \delta)$ such that $f(g(x)) = x$. Also show that for these two functions it is true that $f(g(x)) = g(f(x))$.